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ADVANCED STRENGTH OF MATERIALS

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ADVANCED STRENGTH OF MATERIALS

BY

D. A. R. CLARK
M.Sc.Tech., M.I.Mech.E., A.F.R.Ae.S.
Principal of Constantine Technical College, Middlesbrough

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PREFACE

The book *Materials and Structures* was written to cover Part I of the London University Engineering Degree examination in that subject. Whilst carrying out its original use it was found to be of assistance to students who were studying for the Higher National Diploma and Certificate examinations in Mechanical Engineering as well as for the various Engineering Institutions' examinations. In the past few years I have received many requests from various parts of the world to write a more advanced book. It was felt that the method of presentation was very suitable for the student, but whilst I have in this new book attempted to present the work in a relatively simple manner, there are a few cases in which it is impossible to shorten or simplify the proof. This book, as with the previous one, was built up from lecture work with various grades of students. The method of presentation has been subjected to their criticism over a number of years and in consequence has been remodelled on several occasions. It is now felt that the book *Advanced Strength of Materials* will be of assistance to the student who wishes to take "Strength of Materials" in his final degree examination, particularly London University B.Sc. Engineering Part II, and for the various Higher Diploma and Certificate and Professional Institutions' examinations.

London University have kindly allowed me to use their examination questions and I here acknowledge my indebtedness to them. I record my appreciation of the students who by their questions have enabled me to clarify my thoughts and improve my method of presentation, and of the colleagues who have checked for me the answers of a number of the questions at the end of each chapter. Even after careful checking there would have been a number of errors and unexplained passages except for the meticulous editing by the publishers. I gratefully acknowledge their help and thank them for the great care they have taken in the production of the book. The patience, understanding and encouragement of my wife, who has in addition read through and suggested modifications in the manuscript, have enabled me to complete the work.

D. A. R. CLARK.

MIDDLESBROUGH.

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CHAPTER I

Direct Stress and Strain

1. Problems in which Tension or Area are Varied.

It has been explained in *Materials and Structures* * how simple calculations might be made where the only stress involved is a direct stress. The relationship between stress and strain is known as Hooke's law, which law is accepted as the basis of all work on strength of materials. In the book previously mentioned the cross-sectional area of the component under review has been constant, but it is now necessary to extend our work to examples in which the cross-sectional area is not constant. In such a case the stress at all sections will vary with the cross-sectional area. Whilst the load remains constant there is no advantage in varying the cross-sectional area, but there will be a saving of material in those cases where the load is not constant, e.g. in addition to the external load carried by a pillar there is a stress set up in the material due to the weight of the pillar. The weight carried by any cross-sectional area will depend on the position of that section. The strain will also vary, the total strain being equal to the sum of the individual strains.

Example 1.—A steel bar 15 ft. long is made up of 3 lengths each 5 ft. long. The first length has a cross-sectional area of $1\frac{1}{2}$ sq. in., the second

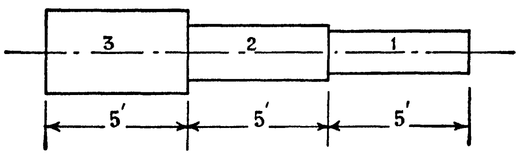


Fig. 1.1

length 2 sq. in., and the third length $2\frac{1}{2}$ sq. in. The bar is subjected to a tensile load of 10 tons. Find the total elongation of the bar. Modulus of elasticity = 12,000 tons per sq. in.

Let p_1 = stress in length 1,

p_2 = stress in length 2,

* *Materials and Structures*: D. A. R. Clark (Blackie).

p_3 = stress in length 3,

A_1 = area of length 1 = $1\frac{1}{2}$ sq. in.,

A_2 = area of length 2 = 2 sq. in.,

A_3 = area of length 3 = $2\frac{1}{2}$ sq. in.,

L_1 = length of 1 = 60 in.,

L_2 = length of 2 = 60 in.,

L_3 = length of 3 = 60 in.,

l_1 = extension of length 1,

l_2 = extension of length 2,

l_3 = extension of length 3,

l = total extension = $l_1 + l_2 + l_3$,

P = load = 10 tons,

$$p_1 = \frac{P}{A_1} = \frac{10}{1\frac{1}{2}} = 6\tfrac{2}{3} \text{ tons per sq. in.,}$$

$$p_2 = \frac{P}{A_2} = \frac{10}{2} = 5 \text{ tons per sq. in.,}$$

$$p_3 = \frac{P}{A_3} = \frac{10}{2\frac{1}{2}} = 4 \text{ tons per sq. in.}$$

From Hooke's law,

stress = strain × modulus of elasticity,

or
$$\frac{P}{A_1} = \frac{l_1}{L_1} \times E.$$

$$\therefore\ l_1 = \frac{PL_1}{A_1E},$$

$$l_2 = \frac{PL_2}{A_2E},$$

and
$$l_3 = \frac{PL_3}{A_3E}.$$

$$\therefore\ l = \frac{PL_1}{A_1E} + \frac{PL_2}{A_2E} + \frac{PL_3}{A_3E}$$

$$= \frac{1}{E}[p_1L_1 + p_2L_2 + p_3L_3]$$

$$= \frac{1}{12{,}000}[6{\cdot}67 \times 60 + 5 \times 60 + 4 \times 60]$$

$$= {\cdot}07835 \text{ in.}$$

There are two possible changes which might take place which will vary the manner of solution of such problems. In the first place, the cross-sectional area might change in such a manner that an algebraic expression might be used to show the change, and secondly, the load on the material might vary along its length. Confining the variation to a change in the load along its length, it may be shown that the rate of increase of the stress is equal to the increase of the load, whilst from Hooke's law the rate of increase of the extension is equal to the stress divided by the modulus of elasticity.

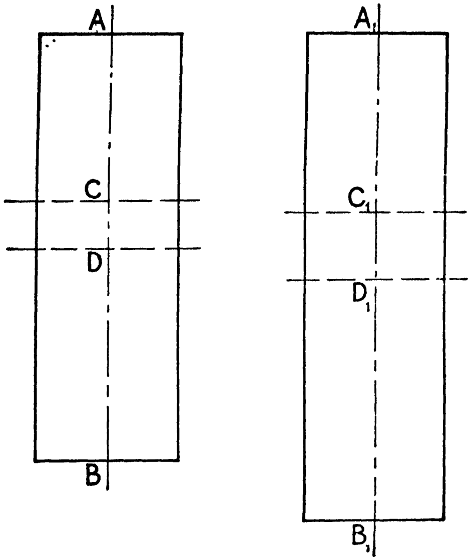


Fig. 1.2

Let AB be an elastic material which is subjected to a continuous load distributed along its length, and A_1B_1 the material after the load has been applied.

If CD is the distance between two points marked before loading, which become C_1D_1 after loading, then the extension of the length CD is $(C_1D_1 - CD)$ and the strain is therefore equal to $\dfrac{C_1D_1 - CD}{CD}$ or $\dfrac{dy}{dx}$.

If T is the total tension at C_1, A the cross-sectional area and E the modulus of elasticity, then, from Hooke's law,

$$\frac{T}{A} = \frac{dy}{dx}E,$$

or

$$T = \frac{dy}{dx}AE. \qquad \qquad (1)$$

If w be the load per unit length at C_1, then the total load is greater at D_1 than at C_1, but this is balanced by the tension being greater at C_1 than D_1. The stress at C_1 will be greater than the stress at D_1. Hence

excess tension at C_1 over that at $D_1 = dT = w\,dx$,

or $$\frac{dT}{dx} = w. \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) may be integrated and consequently the stress and extension at any point may be found. It will be necessary to be given the condition at some known section in order to evaluate the constants of integration. In some problems the load may be a function of x and in consequence would have to be taken into account when integrating.

Example 2.- Let the bar AB be subjected to its own weight, so that w is constant. The known conditions will be that, when $x = L$ where $L =$ AB, the tension $= 0$, and the extension $y = 0$ when $x = 0$.

$$\frac{dT}{dx} = w,$$

or $$T = wx + M.$$

Now $T = 0$ when $x = L$.

$$\therefore\ 0 = wL + M,$$

or $$M = -wL,$$

$$\therefore\ T = wx - wL.$$

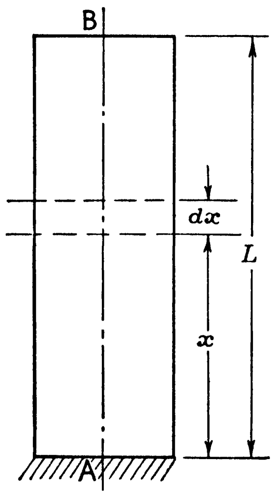


Fig. 1.3

Also $$\frac{dy}{dx} = \frac{T}{AE},$$

$$= \frac{wx - wL}{AE}.$$

$$\therefore\ y = \frac{1}{AE}[\tfrac{1}{2}wx^2 - wLx + N].$$

But $y = 0$ when $x = 0$,

$$\therefore\ N = 0,$$

and $$y = \frac{1}{AE}[\tfrac{1}{2}wx^2 - wLx].$$

$$\text{The tension } T = -wL \text{ when } x = 0,$$

and $$\text{the extension} = \frac{1}{AE}\left[\tfrac{1}{2}wL^2 - wL^2\right]$$

$$= -\frac{wL^2}{2AE} \text{ when } x = L.$$

If there is a load W at the lower end, then the value of the constant M is different, that is,

$$T = W \text{ when } x = L;$$

from which $$W = wL + M,$$

or $$M = W - wL.$$

$$\therefore\ T = wx + (W - wL),$$

and $$\frac{dy}{dx} = \frac{wx + (W - wL)}{AE}.$$

$$\therefore\ y = \frac{1}{AE}\left[\tfrac{1}{2}wx^2 + Wx - wLx + N\right].$$

But $y = 0$ when $x = 0$,

$$\therefore\ N = 0,$$

and $$y = \frac{1}{AE}\left[\tfrac{1}{2}wx^2 + Wx - wLx\right].$$

This gives maximum value of

$$T = (W - wL) \text{ when } x = 0,$$

and $$y = \frac{\tfrac{1}{2}wL^2 + WL - wL^2}{AE} = \frac{-\tfrac{1}{2}wL^2 + WL}{AE} \text{ when } x = L.$$

Example 3.—A hanging column in which the cross-sectional area is varied in order to give uniform strength throughout (the stress in the column being due to its own weight) might be solved in the following manner:

Let the tension at G be T and the tension at H be Q, whilst the area at G is A and the area at H is A_1, then

$$\frac{T}{A} = \frac{Q}{A_1} = f,$$

since the column is of uniform strength.

The difference in the values of the tensions T and Q will be equal to the weight of the small piece of column between G and H, which is $wA\,dx$, so that

$$T - Q = wA\,dx.$$

But $$T = fA$$

and $$Q = fA_1,$$

$$\therefore fA - fA_1 = wA\,dx,$$

or $$f(A - A_1) = wA\,dx.$$

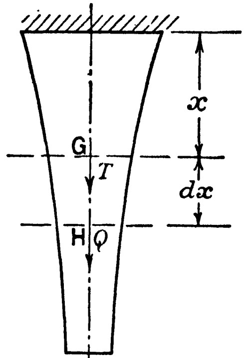


Fig. 1.4

If $$A_1 = A + dA,$$

then $$A - A_1 = -dA,$$

and $$-f\,dA = wA\,dx,$$

or $$\frac{dA}{A} = -\frac{w}{f}\,dx.$$

Integrating $$\int \frac{dA}{A} = -\int \frac{w}{f}\,dx.$$

$$\therefore \log_e A = -\frac{wx}{f} + C,$$

or
$$\begin{aligned} A &= e^{C - wx/f} \\ &= e^C \times e^{-wx/f} \\ &= De^{-wx/f}. \end{aligned}$$

If the area be A_0 when $x = 0$, then

$$A_0 = D,$$

and the area at any section distant x from the top is

$$A = A_0 e^{-wx/f}.$$

Thus the tension at any section distant x from the top is

$$fA_0 e^{-wx/f}.$$

From Hooke's law, $$f = \frac{dy}{dx} E,$$

since the stress is constant, or

$$\frac{dy}{dx} = \frac{f}{E}.$$

$$\therefore y = \frac{fx}{E} + F.$$

When $x = 0$, $y = 0$, $\therefore F = 0.$

$$\therefore y = \frac{fx}{E}.$$

2. Stress due to Centrifugal Action.

If a bar of uniform cross-section A and length L be rotated with an angular velocity ω, then the stress on any section will vary.

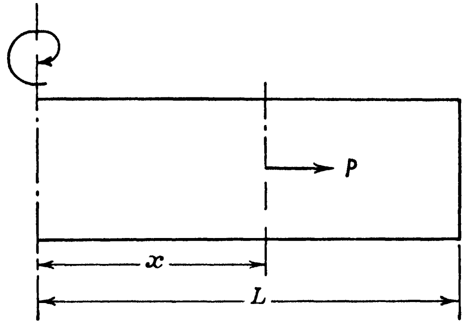


Fig 1.5

Weight of element at C $= dW = A\,dx\,w$.
Centrifugal force at C due to dW is

$$dP = \frac{dW\,\omega^2 x}{g} = \frac{A\,dx\,.\,w\omega^2 x}{g}.$$

Total tension at C due to centrifugal force

$$P = \int_x^L \frac{A\,dx\,.\,w\omega^2 x}{g} = \frac{Aw\omega^2}{g}\left[\frac{x^2}{2}\right]_x^L$$

$$= \frac{Aw\omega^2}{g}\left[\frac{L^2 - x^2}{2}\right].$$

When $x = 0$, $$P = \frac{Aw\omega^2}{g}\,\frac{L^2}{2}.$$

From Hooke's law, $$\frac{P}{A} = \frac{dy}{dx} \times E,$$

or $$\frac{Aw\omega^2}{Ag}\left[\frac{L^2 - x^2}{2}\right] = \frac{dy}{dx}E,$$

i.e. $$\frac{dy}{dx} = \frac{w\omega^2}{2gE}(L^2 - x^2),$$

$$\therefore\ y = \int_x^L \frac{w\omega^2}{2gE}(L^2 - x^2)\,dx$$

$$= \frac{w\omega^2}{2gE}\left(\tfrac{2}{3}L^3 - L^2 x + \tfrac{1}{3}x^3\right).$$

When $x = 0$,

$$y = \frac{w\omega^2}{2gE}\,\tfrac{2}{3}L^3$$

$$= \tfrac{1}{3}\frac{w\omega^2}{gE}\,L^3.$$

Example 4.—A round steel rod $1\frac{1}{2}$ in. in diameter and 36 in. long is rotated about a fixed axis which passes through one end of the rod and at right angles to its length. The speed of rotation is 1200 revolutions per minute, and the weight of the material is 0·28 lb. per c. in. If the modulus of elasticity of the material be 30×10^6 lb. per sq. in., find the total elongation of the rod, and the total stresses at sections 0, 9, 18 and 27 in. from the axis of rotation.

Use $$P = \frac{Aw\omega^2}{g}\left[\frac{L^2 - x^2}{2}\right],$$

where $$A = \frac{\pi}{4} \times \left(\frac{3}{2}\right)^2 = 1{\cdot}766 \text{ sq. in.},$$

$$w = 0{\cdot}28 \text{ lb. per c. in.},$$

$$\omega = \frac{1200 \times 2\pi}{60} = 125{\cdot}6 \text{ radians per sec.},$$

$$g = 32{\cdot}2 \times 12 = 386{\cdot}4 \text{ in. per sec. per sec.}$$

When $x = 0$,

$$P = \frac{1{\cdot}766 \times {\cdot}28 \times 125{\cdot}6^2}{386{\cdot}4} \times \frac{36^2}{2}$$

$$= 13{,}080 \text{ lb.}$$

When $x = 9$ in.,

$$P = \frac{1{\cdot}766 \times {\cdot}28 \times 125{\cdot}6^2}{386{\cdot}4}\left[\frac{36^2 - 9^2}{2}\right]$$

$$= 10{\cdot}09 \times 36^2\left[1 - \frac{9^2}{36^2}\right].$$

$$= 12{,}260 \text{ lb.}$$

When $x = 18$ in.,

$$P = \frac{1{\cdot}766 \times {\cdot}28 \times 125{\cdot}6^2}{386{\cdot}4}\left[\frac{36^2 - 18^2}{2}\right]$$

$$= 10{\cdot}09 \times 36^2\left[1 - \frac{18^2}{36^2}\right]$$

$$= 13{,}080 \times \tfrac{3}{4}$$

$$= 9810 \text{ lb.}$$

When $x = 27$ in.,

$$P = 13{,}080\left[1 - \frac{27^2}{36^2}\right]$$

$$= 5723 \text{ lb.}$$

The total elongation may be found by substituting in the equation

$$y = \tfrac{1}{3}\frac{w\omega^2}{gE}L^3$$

$$= \tfrac{1}{3} \times \frac{\cdot 28 \times 125{\cdot}6^2 \times 36^3}{386{\cdot}4 \times 30 \times 10^6}$$

$$= \cdot 00593 \text{ in.}$$

Example 5.—A conical steel rod is 24 in. long and is rotated about a fixed axis which passes through the large end and at right angles to its length. The area of the large end is 2 sq. in. Determine the maximum

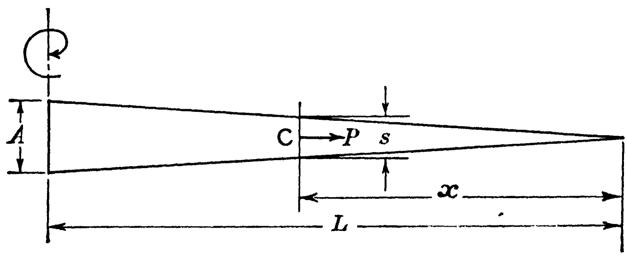


Fig. 1.6

tension and the total elongation of the rod. Density of material is 0·28 lb. per c. in., modulus of elasticity 30×10^6 lb. per sq. in., and the number of revolutions per minute 3000.

Weight of element at C is $dW = s\,dx\,w$, where

s = cross-sectional area at C,

ω = angular velocity,

w = density of material.

Centrifugal force at C due to dW is

$$dP = \frac{dW \,.\, \omega^2(L - x)}{g} = \frac{s\,dx\,w\omega^2(L - x)}{g}.$$

Tension at C due to centrifugal force is

$$P = \int_0^x \frac{s\,dx\,w\omega^2}{g}(L - x).$$

Before integrating it is necessary to state s in terms of fixed dimensions, as the area varies with the position of the section.

From similar figures

$$\frac{s}{x^2} = \frac{A}{L^2}.$$

$$\therefore s = \frac{Ax^2}{L^2}.$$

Substituting,

$$P = \int_0^x \frac{Ax^2}{L^2}\,dx\,\frac{w\omega^2(L - x)}{g}$$

$$= \frac{Aw\omega^2}{gL^2}\int_0^x (L - x)x^2\,dx$$

$$= \frac{Aw\omega^2}{gL^2}\left[\tfrac{1}{3}Lx^3 - \tfrac{1}{4}x^4\right]_0^x$$

$$= \frac{Aw\omega^2}{gL^2}\left[\tfrac{1}{3}Lx^3 - \tfrac{1}{4}x^4\right].$$

The maximum tension will occur when $x = L$, then

$$P = \frac{Aw\omega^2}{gL^2}\left[\tfrac{1}{3}L^4 - \tfrac{1}{4}L^4\right]$$

$$= \tfrac{1}{12}\frac{Aw\omega^2L^4}{gL^2}$$

$$= \tfrac{1}{12}\frac{Aw\omega^2L^2}{g}.$$

When $w = 0{\cdot}28$ lb. per c. in.,

$L = 24$ in.,

$A = 2$ sq. in.,

$g = 32{\cdot}2 \times 12 = 386{\cdot}4$ in. per sec. per sec.,

$\omega = \frac{3000}{60} \times 2\pi = 314$ radians per sec.,

$$P = \tfrac{1}{12}\,\frac{2 \times {\cdot}28 \times 314^2 \times 24^2}{386{\cdot}4}$$

$$= 6857 \text{ lb.}$$

From Hooke's law,

$$\frac{\text{tension at C}}{\text{area at C}} = \frac{dy}{dx}E.$$

$$\therefore \frac{\frac{Aw\omega^2}{gL^2}\left[\frac{1}{3}Lx^3 - \frac{1}{4}x^4\right]}{s} = \frac{dy}{dx}E,$$

or

$$\frac{dy}{dx} = \frac{\frac{Aw\omega^2}{EgL^2}\left[\frac{1}{3}Lx^3 - \frac{1}{4}x^4\right]}{\frac{Ax^2}{L^2}}.$$

$$\therefore y = \frac{w\omega^2}{Eg}\int_0^x \frac{\left(\frac{1}{3}Lx^3 - \frac{1}{4}x^4\right)dx}{x^2}$$

$$= \frac{w\omega^2}{Eg}\left[\frac{Lx^2}{6} - \frac{x^3}{12}\right]_0^x$$

$$= \frac{w\omega^2}{Eg}\left[\frac{Lx^2}{6} - \frac{x^3}{12}\right].$$

This will be a maximum when $x = L$, then

$$y = \frac{w\omega^2}{Eg}\left[\frac{L^3}{6} - \frac{L^3}{12}\right]$$

$$= \frac{w\omega^2 L^3}{12Eg}.$$

Substituting values,

$$y = \frac{\cdot 28 \times 314^2 \times 24^3}{12 \times 30 \times 10^6 \times 386\cdot 4}$$

$$= \cdot 002743 \text{ in.}$$

When the cross-sectional area varies in a manner which cannot be expressed in an algebraic form it will be necessary to integrate graphically. This is the case with propellers. In addition to the tensile stress caused by the centrifugal force there will be shearing and bending stresses acting on the material. In the following example the centrifugal force only is considered.

Example 6.—An aeroplane propeller is made of mahogany which has a density of 35 lb. per c. ft.; it rotates at 1500 r.p.m. and is 10 ft. in diameter. The cross-sectional area, A sq. in., is given below for different distances, r in., from the axis of rotation. Draw a curve (a) to show the variation of stress along the blade, (b) to show the total centrifugal force

variation along the blade, and (c) calculate the total extension of the blade. Modulus of elasticity of mahogany = $1{\cdot}5 \times 10^6$ lb. per sq. in.

r in.:	6	12	18	24	30	36	42	48	54	57	60.
A sq. in.:	26·7	18·67	15·0	12·6	11·0	9·65	8·4	7·24	5·95	4·91	0.

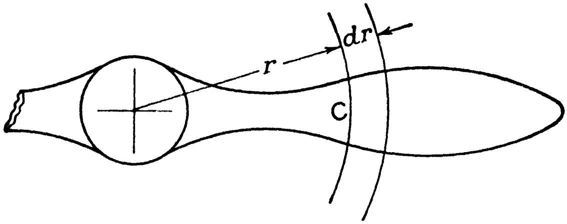


Fig. 1.7

Weight of element at C is $dW = A\,dr w$.

$$\text{Centrifugal force at C} = dP = \frac{dW\omega^2 r}{g}$$

$$= \frac{A\,dr\,w\omega^2 r}{g}.$$

Total tension at C due to centrifugal force $P = \int_r^L \frac{A\,dr w\omega^2 r}{g}$, where

A = area in square inches,

w = density = $\frac{35}{1728}$ lb. per c. in.,

ω = angular velocity = $\frac{1500}{60} \times 2\pi$ = 157 radians per sec.,

$g = 32{\cdot}2 \times 12 = 386{\cdot}4$ in. per sec. per sec.,

L = length = 60 in.

$$\therefore\ P = \int_r^{60} A\,dr \times \frac{35}{1728} \times \frac{157^2}{386{\cdot}4}\,r$$

$$= \int_r^{60} 1{\cdot}295\,Ar\,dr.$$

As the area varies with the radius, but not in a manner which can be expressed algebraically, it will be necessary to integrate graphically.

Plot the product Ar vertically and r horizontally, then the area under the curve within the limits r and 60 will give the total tension at the section at radius r when multiplied by 1·295.

r in.	*Ar* in.^{3}	$\int_r^{60} Ar\,dr$	$1{\cdot}295\int_r^{60} Ar\,dr$ (lb.)	Stress (lb. per sq. in.)
6	160	16,100	20,800	780
12	224	15,000	19,400	1040
18	270	13,500	17,500	1168
24	302 5	11,800	15,300	1220
30	330	9,870	12,800	1160
36	347	7,830	10,140	1050
42	353	5,700	7,360	875
48	347	3,590	4,650	643
54	321	1,570	2,030	342
57	280	670	860	176
60	0	0	0	0

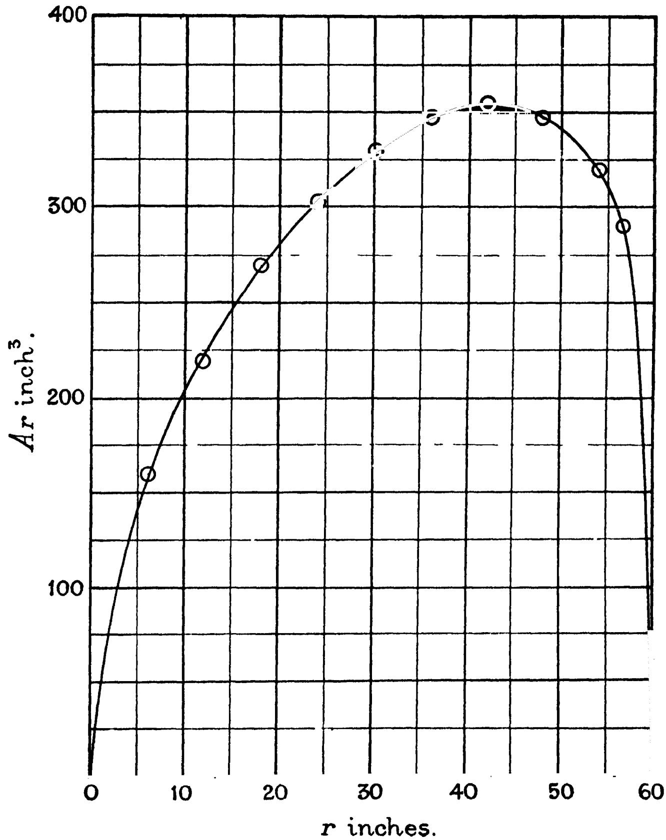


Fig 1.8

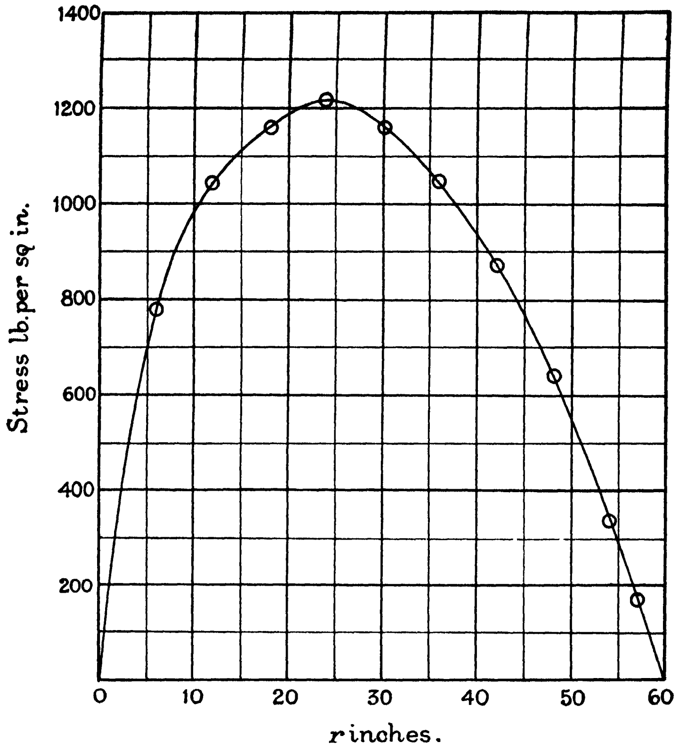


Fig. 1.9

From Hooke's law, stress $= \dfrac{dy}{dx} E.$

$$\therefore\ dy = \frac{1}{E}\,(\text{stress})\,dx.$$

$$\therefore\ y = \frac{1}{E}\int_r^L (\text{stress})\,dx.$$

It is therefore necessary to plot the stress curve and integrate it in order to find the extension.

Hence the total extension $= \dfrac{1}{E}\displaystyle\int_0^{60} (\text{stress})\,dx$

$$= \frac{49{,}900}{E}$$

$$= \frac{49{,}900}{1{\cdot}5 \times 10^6}$$

$$= {\cdot}033 \text{ in.}$$

3. Effects produced by Unequal Expansion.

Whilst compound bars subjected to tensile or compressive stresses, and single bars subjected to stresses due to temperature changes without allowing the bar to change its original length, are comparatively straightforward, problems which combine both effects are more complicated. Providing the problem is taken from first principles, the solution may be obtained in a straightforward manner. In order to illustrate the method of solution the following example is taken as being typical of its type.

Example 7.—A steel bolt of $\frac{3}{4}$ in. diameter passes through a brass tube of $\frac{3}{4}$ in. internal and $1\frac{1}{4}$ in. external diameter. When both are at a uniform temperature of 50° F., the nut on the bolt is tightened until there is a stress of 3200 lb. per sq. in. in the bolt. The tube and bolt are then heated to a uniform temperature of 250° F. What are then the stresses in the bolt and the tube?

Coefficient of expansion of brass $10{\cdot}5 \times 10^{-6}$ per degree F.,

coefficient of expansion of steel $= 7{\cdot}0 \times 10^{-6}$ per degree F.,

Young's modulus for brass $= 12 \times 10^{6}$ lb. per sq. in.,

Young's modulus for steel $= 30 \times 10^{6}$ lb. per sq. in.

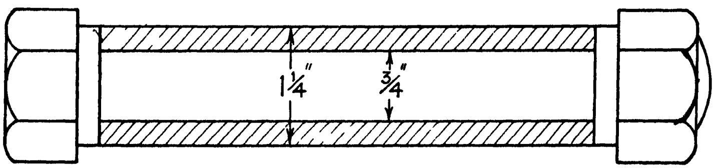


Fig. 1.10

The initial stress in the bolt will be tensile. The magnitude of the total stress in the bolt will be

$$\tfrac{1}{4}\pi(\tfrac{3}{4})^2 \times 3200 - 1413 \text{ lb.}$$

Since the system is in equilibrium, then the total stress in the brass will be 1413 lb. and will be compressive.

When the system is heated, both the bolt and the tube will expand. If the coefficient of expansion for brass and steel were the same, then the stresses would remain the same irrespective of the temperature. But as the coefficient for brass is greater than the coefficient for steel, the brass will tend to expand more than the steel. If the steel and brass were not connected the increases in length would be $L_s\alpha_s t$ and $L_b\alpha_b t$ respectively, where L_s and L_b are the original lengths of steel and brass respectively, α_s and α_b are coefficients for steel and brass respectively, and t is the rise in temperature.

$$\text{Extension of brass} = L_b \times 10{\cdot}5 \times 10^{-6} \times 200$$
$$= 21 \times 10^{-4} L_b = l_b.$$
$$\text{Extension of steel} = L_s \times 7 \times 10^{-6} \times 200$$
$$= 14 \times 10^{-4} L_s = l_s.$$

But $$L_b = L_s = L.$$

The final length of the compound system must be as long as $(L + 14 \times 10^{-4} \times L)$, since both the steel and the brass extend to that length, but cannot be longer than $(L + 21 \times 10^{-4} \times L)$, since neither material extends beyond that length. Hence the final length will lie between $(L + 14 \times 10^{-4}L)$ and $(L + 21 \times 10^{-4}L)$.

Let the final length be $L + x$, where x lies between $14 \times 10^{-4}L$ and $21 \times 10^{-4}L$. If x is greater than $14 \times 10^{-4}L$, then there will be an increase in length of the steel of $(x - 14 \times 10^{-4}L)$ and a shortening of the brass of $(21 \times 10^{-4}L - x)$.

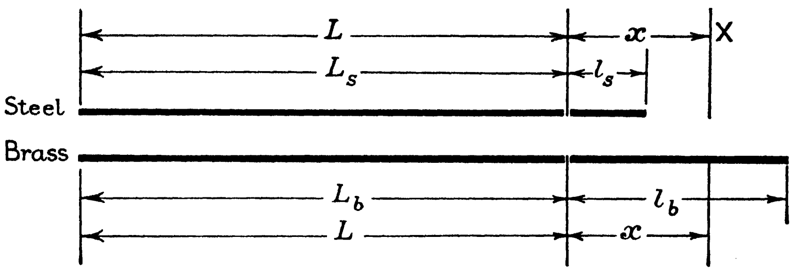


Fig. 1.11

Let l_s be extension of steel, and l_b the shortening of brass, then

$$l_s + l_b = 21 \times 10^{-4}L - 14 \times 10^{-4}L$$
$$= 7 \times 10^{-4}L. \quad \text{. . . . . (i)}$$

Since the final system is in equilibrium, the total tensile force in the steel must balance the total compressive force in the brass, i.e.

$$P_s = P_b,$$

where P_s and P_b are the total forces in steel and brass respectively.

But $$\frac{P_s}{a_s} = \frac{l_s}{L} E_s,$$

and $$\frac{P_b}{a_b} = \frac{l_b}{L} E_b,$$

where a_s and a_b are the areas of steel and brass, and E_s and E_b are the moduli of elasticity of steel and brass respectively.

Now
$$\frac{P_s}{\frac{\pi}{4} \times \left(\frac{3}{4}\right)^2} = \frac{l_s}{L} \times 30 \times 10^6,$$

or
$$P_s = \cdot 4418 \frac{l_s}{L} \times 30 \times 10^6, \quad . \quad . \quad . \quad \text{(ii)}$$

and
$$\frac{P_b}{\frac{\pi}{4}\left[(1\frac{1}{4})^2 - (\frac{3}{4})^2\right]} = \frac{l_b}{L} \times 12 \times 10^6,$$

or
$$P_b = \cdot 7854 \frac{l_b}{L} \times 12 \times 10^6.$$

But
$$P_s = P_b,$$

$$\therefore \cdot 4418 \frac{l_s}{L} \times 30 \times 10^6 = \cdot 7854 \frac{l_b}{L} \times 12 \times 10^6.$$

$$\therefore 13{\cdot}254 l_s = 9{\cdot}4248 l_b.$$

$$\therefore l_b = \frac{13{\cdot}254}{9{\cdot}4248} l_s$$

$$= 1{\cdot}406 l_s.$$

But
$$l_s + l_b = 7 \times 10^{-4} L, \qquad \text{(equation i)}$$

$$\therefore l_s + 1{\cdot}406 l_s = 7 \times 10^{-4} L,$$

or
$$l_s = 2{\cdot}909 L \times 10^{-4}.$$

Now
$$\frac{P_s}{\cdot 4418} = \frac{l_s}{L} \times 30 \times 10^6, \qquad \text{(equation ii)}$$

$$\therefore P_s = \cdot 4418 \times 2{\cdot}909 \times 10^{-4} \times 30 \times 10^6$$

$$= 3855 \text{ lb.}$$

There is an initial total tensile load in the steel of 1413 lb., therefore the total tensile load in the steel = 3855 + 1413 = 5268 lb. Hence tensile stress = $\frac{5268}{\cdot 4418}$ = 11,920 lb. per sq. in. There is an initial total compressive load in the brass of 1413 lb., therefore final total compressive load in the brass = 5268 lb., and the compressive stress in the brass = $\frac{5268}{\cdot 7854}$ = 6708 lb. per sq. in.

4. Self-straining Problem.

A further example of this self-straining effect is shown when an elastic packing piece is placed between two rigid plates which are

bolted together; such an example often takes place when a machine is bolted to its foundation and an elastic packing piece inserted between the machine and foundation. If one of the nuts is slightly slackened off, the packing piece will continue to impose a tensile stress on the bolts.

Example 8.—An elastic packing piece is bolted between a rigid rectangular plate and a rigid foundation by two bolts pitched 10 in. apart and symmetrically placed on the longer centre of the plate, which is 15 in.

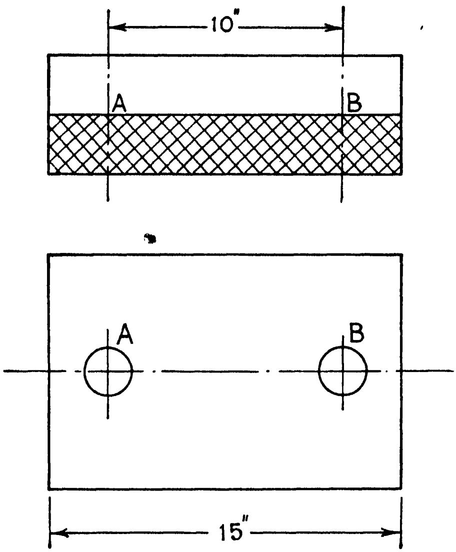


Fig. 1.12

long. The tension in each bolt is initially 4000 lb., the extension of each bolt is 0·0005 in., and the compression of the packing piece is 0·02 in. If one bolt is further tightened to a tension of 5000 lb., determine the tension in the other bolt.

If the bolt A is tightened, the.. the packing piece takes up the shape shown in fig. 1.13. The compressions of the packing piece at bolt A and B are e_a and e_b respectively—after bolt A is tightened.

The compression in the packing piece at XX = z

$$= e_b + (e_a - e_b) \cdot \frac{5 + x}{10}$$

$$= \tfrac{1}{2}(e_a + e_b) + \tfrac{1}{10}(e_a - e_b)x.$$

From Hooke's law,

total stress $\propto$ compression $= K \times$ compression.

$$\therefore \text{ Total force on packing piece } = K\int_{-7\frac{1}{2}}^{+7\frac{1}{2}} [\tfrac{1}{2}(e_a + e_b) + \tfrac{1}{10}(e_a - e_b)x]\, dx.$$

$$= K[\tfrac{1}{2}(e_a + e_b)x + \tfrac{1}{20}(e_a - e_b)x^2]_{-7\frac{1}{2}}^{+7\frac{1}{2}}$$

$$= K(e_a + e_b) \times 7\tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

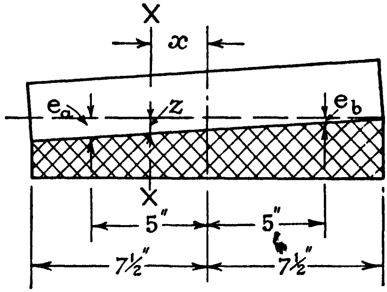


Fig 1 13

If e_1 be the compression of the packing piece before bolt A is further tightened, then the total force on the packing piece $= Ke_1 \times 15$. But the total force equals the total tensions in the bolt $= 2T$.

$$\therefore\ Ke_1 \times 15 = 2T;$$

$$\therefore\ K = \frac{2T}{15e_1}. \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Substituting for K in equation (i),

$$\text{total force on packing piece} = \frac{T}{e_1}(e_a + e_b).$$

The moment of the force on the packing piece about the centre line

$$= K\int_{-7\frac{1}{2}}^{+7\frac{1}{2}} [\tfrac{1}{2}(e_a + e_b) + \tfrac{1}{10}(e_a - e_b)x]x\, dx$$

$$= K[\tfrac{1}{4}(e_a + e_b)x^2 + \tfrac{1}{30}(e_a - e_b)x^3]_{-7\frac{1}{2}}^{+7\frac{1}{2}}$$

$$= K\tfrac{1}{15}(e_a - e_b)(7\tfrac{1}{2})^3.$$

Substituting for K from (ii),

$$\text{moment} = 3\tfrac{3}{4}T\frac{(e_a - e_b)}{e_1}.$$

Let P be the final total load on the bolt B. Then

$$P + 5000 = \text{total force on packing piece}$$
$$= \frac{T}{e_1}(e_a + e_b).$$
$$\therefore\ e_a + e_b = (P + 5000)\frac{e_1}{T}$$

(where $e_1 = 0{\cdot}02$ in. and $T = 4000$ lb.)

$$= (P + 5000) \times \frac{\cdot 02}{4000}$$
$$= \frac{P + 5000}{200{,}000}. \quad \text{(iii)}$$

Taking moments about the centre line of the total forces in the bolts,

$$5000 \times 5 - P \times 5 = \text{moment of force on packing piece}$$
$$= 3\tfrac{3}{4}\frac{T}{e_1}(e_a - e_b).$$
$$\therefore\ e_a - e_b = (25{,}000 - 5P)\frac{e_1}{3\tfrac{3}{4}T}$$
$$= \frac{(25{,}000 - 5P) \times \cdot 02}{3\tfrac{3}{4} \times 4000}$$
$$= \frac{25{,}000 - 5P}{750{,}000}. \quad \text{(iv)}$$

Subtracting equation (iv) from (iii),

$$2e_b = \frac{P + 5000}{200{,}000} - \frac{25{,}000 - 5P}{750{,}000}$$
$$= \frac{35P - 25{,}000}{15 \times 200{,}000},$$
$$\therefore\ e_b = \frac{7P - 5000}{600{,}000 \times 2}.$$

The reduction in the compression of the packing piece at A will be equal to the increase in the extension of the bolt B, i.e. $e_1 - e_b$. When the total tensile load on the bolts is 4000 lb., the extension is ·0005 in., therefore an increase of extension of $(e_1 - e_b)$ will result in an increase of total load equal to $(e_1 - e_b)\,4000 \div 0{\cdot}0005$.

But increase of total stress $= P - 4000$,

$$\therefore P - 4000 = (e_1 - e_b)\frac{4000}{\cdot 0005}$$

$$= (\cdot 02 - e_b)\frac{4000}{\cdot 0005}$$

$$= \left[\cdot 02 - \frac{(7P - 5000)}{2 \times 600{,}000}\right]\frac{4000}{\cdot 0005}$$

$$= 160{,}000 + \frac{100{,}000}{3} - \frac{140P}{3}.$$

$$\therefore \frac{143P}{3} = \frac{592{,}000}{3}$$

$$\therefore P = 4140 \text{ lb.}$$

EXAMPLES

1. A tube of brass, 1 in. internal diameter, $1\frac{1}{2}$ in. external diameter, is fitted over a steel bar 1 in. diameter of the same length, so that both act as one bar. An axial compressive load of 6 tons is applied to the compound bar. What is the stress in each material?

Whilst still under this load the temperature of the bar is raised by 50° F. What is then the stress in each material? For brass, $E = 6500$ tons per sq. in.; coefficient of linear expansion $= 0{\cdot}00001$ per °F. For steel, $E = 13{,}000$ tons per sq. in.; coefficient of linear expansion $= 0{\cdot}0000066$ per °F.

[Steel, 4·7 tons per sq. in.; brass, 2·35 tons per sq. in : steel, 3·4 tons per sq. in.; brass, 3·39 tons per sq. in.]

2. A copper tube and a steel tube of equal areas of cross-section are clamped together between two steel end-pieces by four Invar bolts. The area of cross-section of each bolt is one-sixteenth that of either tube. Find the change in stress in each of the tubes and bolts for a rise of temperature of one degree. Neglect the strain in the end-pieces due to stress. E for steel tube and bolts $= 30 \times 10^6$ lb. per sq. in. E for copper tube $= 15 \times 10^6$ lb. per sq. in. Coefficients of linear expansion: copper $= 18 \times 10^{-6}$ per °C., steel $= 11 \times 10^{-6}$ per °C., Invar—nil.

[Steel, 12·9 lb. per sq. in. tension; copper, 83·55 lb. per sq. in. compression; Invar, 343 lb. per sq. in. tension.]

3. A steel tie-rod 1 in. in diameter is placed concentrically in a brass tube $\frac{1}{8}$ in. thick and $2\frac{1}{2}$ in. mean diameter. Nuts and washers are fitted on the tie-rod so that the ends of the tube are enclosed by the washers. The nuts are initially tightened to give a compressive stress of 2 tons per sq. in. in the tube, and a tensile load of $4\frac{1}{2}$ tons is then applied to the tie-rod. Assuming the rod and tube to have the same effective length, find the

resultant stresses in the tie-rod and tube: (i) when there is no change of temperature, (ii) when the temperature increases by 60° C.

E for steel $= 30 \times 10^6$ lb. per sq. in.; E for brass $= 12 \times 10^6$ lb. per sq. in. Coefficient of expansion for steel $= 1{\cdot}1 \times 10^{-5}$ per °C.; coefficient of expansion for brass $= 1{\cdot}89 \times 10^{-5}$ per °C. (Univ. of London.)

[(i) Bolt, 6·32 tons per sq. in. tension; tube, 0·47 ton per sq. in. compression. (ii) Bolt, 8·44 tons per sq. in. tension; tube, 2·165 tons per sq. in. compression.]

4. A compound bar is made of one bar of steel and two bars of copper. The bars are all of the same size and are fixed together at 60° F. When the temperature of the compound bar has been raised to 560° F., the length of the bar has been increased by 0·05 in. Find the stress in the steel and copper bars at this temperature. What was the original length of the compound bar? E for copper $= 7000$ tons per sq. in.; E for steel $= 14{,}000$ tons per sq. in. Coefficients of linear expansion for copper and steel respectively are 0·00001 and 0·0000069.

[Steel, 10·8 tons per sq. in.; copper, 5·4 tons per sq. in.; 11·84 ft.]

5. A steel tube 20 in. long, 1 in. internal diameter and $1\frac{1}{2}$ in. external diameter is surrounded by a copper tube $1\frac{1}{2}$ in. internal and 2 in. external diameter, the two being securely fastened to each other to prevent any relative axial movement. This unit is now subjected to a rise of 200° C. in temperature. While at this high temperature a steel rod $\frac{3}{4}$ in. diameter is slipped through the inner tube, and by means of nuts on the end, the first unit is put in compression; to do this the nuts are made just tight, and then given another $\frac{1}{4}$ turn relative to one another. The $\frac{3}{4}$ in. rod is threaded 10 threads per inch. Find the final stress in the copper tube before any further changes of temperature occur. Coefficient of expansion for steel $= 1{\cdot}0 \times 10^{-5}$ per °C., coefficient of expansion for copper $= 1{\cdot}5 \times 10^{-5}$ per °C., Young's modulus for steel $= 30 \times 10^6$ lb. per sq. in., Young's modulus for copper $= 15 \times 10^6$ lb. per sq. in. [12,800 lb. per sq. in.]

6. An elastic packing piece is bolted between a rigid rectangular plate and a rigid foundation by two bolts 8 in. apart, symmetrically placed on the centre-line of the plate, which is 12 in. long. The tension in each bolt is 4000 lb., the extension of each bolt is 0·0008 in., and the compression of the packing piece is 0·04 in. Determine the magnitude of the vertical load which when applied on the line of the bolts at the end of the plate will just reduce to zero the tension in one bolt. (Univ. of London.)

[5420 lb.]

7. Two equal elastic washers are compressed between a rigid base-plate and a rigid bridge-piece by two bolts screwed into the base-plate and passing through clearance holes in the bridge-piece. The centre-lines of the bolts and washers are on the longitudinal centre-line of the bridge-piece; the washers are 8 in. apart and the bolts 12 in. apart, placed symmetrically on either side of the washers. The bolts are each initially tightened to a tension of 6000 lb., and the extension of each is 0·002 in. If the deformation of a washer is three times that of a bolt for the same load, find the tension in the other bolt when one is further tightened to a tension of 8000 lb. (Univ. of London.) [6222 lb.]

8. Two equal elastic washers, 6 in. apart, are compressed between a rigid horizontal base and a rigid horizontal plate by two equal bolts. The bolts are 12 in. apart, arranged symmetrically on either side of the washers and collinear with them. The bolts pass through clearance holes in the plate and initially each bolt is tightened to a tension of 6000 lb. with extension of 0·0018 in. If the compression of a washer is four times the extension of a bolt for the same value of the load, determine the increase of tension in one bolt when the other is further tightened to a tension of 8000 lb. (Univ. of London.) [383 lb.]

CHAPTER II

Beams

1. Relationship between Load, Shearing Force and Bending Moment.

The relationship between load intensity, shearing force and bending moment has been shown in *Materials and Structures*. The relationship which was obtained might be stated mathematically as

$$\text{S.F.} = \int (\text{load intensity}).$$

$$\begin{aligned} \text{B.M.} &= \int \text{S.F.} \\ &= \iint (\text{load intensity}). \end{aligned}$$

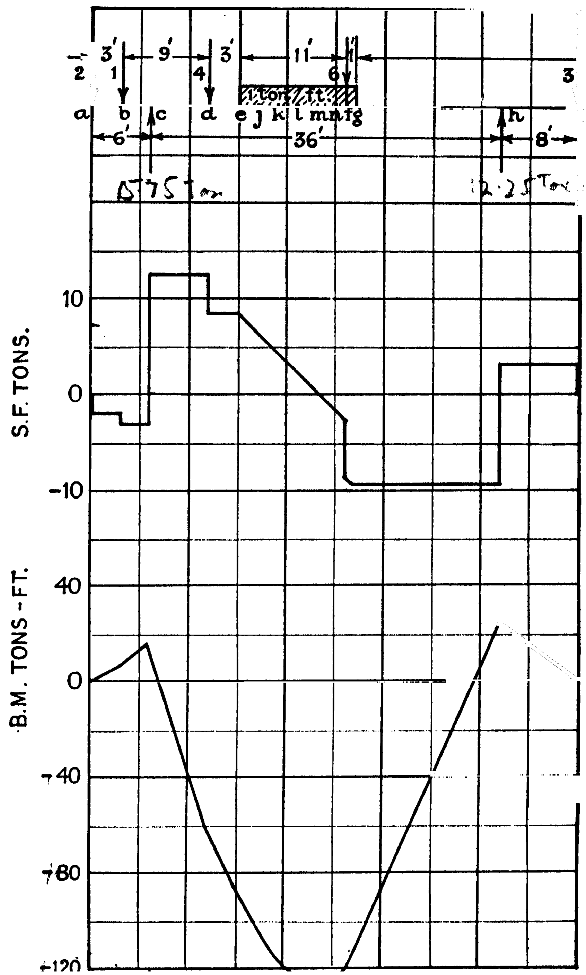
It might be necessary to remind the student that the above mathematical solution is only applicable when the load is distributed in such a way that an algebraic expression can fully explain it. For loading which contains concentrated loads, it will be necessary to calculate the shearing force and bending moment at particular sections of the beam. In order to explain the latter method the following example is fully worked out.

Example 1.—A beam of 50 ft. length is simply supported over a span of 36 ft. at such positions that there is an overhang of 6 ft. at the left end and an overhang of 8 ft. at the right end. There are concentrated loads of 2, 1, 4, 6 and 3 tons at 0, 3, 12, 26 and 50 ft. from the left support, and a uniformly distributed load of 1 ton per foot over a 12 ft. length of the beam starting at a point 15 ft. from the left support. Calculate the bending moments and shearing forces at the principal sections of the beam, and draw the bending-moment and shearing-force diagrams.

The reactions at c and h must be found before the shearing forces and bending moments can be calculated.

Moments about h:

$$\begin{aligned} R_c \times 36 + 3 \times 8 &= 2 \times 42 + 1 \times 39 + 4 \times 30 + 6 \times 16 + 12 \times 21 \\ &= 84 + 39 + 120 + 96 + 252; \\ \therefore\ 36R_c &= 591 - 24 \\ &= 567, \\ \therefore\ R_c &= 15\tfrac{3}{4} \text{ tons.} \\ R_h &= 28 - 15\tfrac{3}{4} = 12\tfrac{1}{4} \text{ tons.} \end{aligned}$$



Shearing force a to b = -2 tons.

$$\text{S.F.}_{bc} = -(2 + 1) = -3 \text{ tons.}$$

$$\text{S.F.}_{cd} = -(2 + 1) + 15\tfrac{3}{4} = +12\tfrac{3}{4} \text{ tons.}$$

$$\text{S.F.}_{de} = -(2 + 1) + 15\tfrac{3}{4} - 4 = 8\tfrac{3}{4} \text{ tons.}$$

Taking points j, k, l, m, n, at 2 ft. intervals along the distributed load, the shearing forces at these points may be calculated.

$$S.F._{j} = S.F._{e} - 2 = 6\tfrac{3}{4} \text{ tons.}$$
$$S.F._{k} = S.F._{j} - 2 = 4\tfrac{3}{4} \text{ tons.}$$
$$S.F._{l} = S.F._{k} - 2 = 2\tfrac{3}{4} \text{ tons.}$$
$$S.F._{m} = S.F._{l} - 2 = \tfrac{3}{4} \text{ ton.}$$
$$S.F._{n} = S.F._{m} - 2 = -1\tfrac{1}{4} \text{ tons.}$$
$$S.F._{f} \text{ (right of load)} = S.F._{n} - 1 - 6 = -8\tfrac{1}{4} \text{ tons.}$$
$$S.F._{g} = S.F._{f} - 1 = -9\tfrac{1}{4} \text{ tons.}$$
$$S.F._{gh} = S.F._{g} = -9\tfrac{1}{4} \text{ tons.}$$
$$S.F._{hi} = S.F._{gh} + 12\tfrac{1}{4} = +3 \text{ tons.}$$

Bending moment at $a = 0$.

$$B.M._{b} = 2 \times 3 = 6 \text{ tons-ft.}$$
$$B.M._{c} = 2 \times 6 + 1 \times 3 = 15 \text{ tons-ft.}$$
$$B.M._{d} = 2 \times 12 + 1 \times 9 - 15\tfrac{3}{4} \times 6$$
$$= -61\tfrac{1}{2} \text{ tons-ft.}$$
$$B.M._{e} = 2 \times 15 + 1 \times 12 + 4 \times 3 - 15\tfrac{3}{4} \times 9$$
$$= -87\tfrac{3}{4} \text{ tons-ft.}$$
$$B.M._{j} = 2 \times 17 + 1 \times 14 + 4 \times 5 + 2 \times 1 - 15\tfrac{3}{4} \times 11$$
$$= -103\tfrac{1}{4} \text{ tons-ft.}$$
$$B.M._{k} = 2 \times 19 + 1 \times 16 + 4 \times 7 + 4 \times 2 - 15\tfrac{3}{4} \times 13$$
$$= -114\tfrac{3}{4} \text{ tons-ft.}$$
$$B.M._{l} = 2 \times 21 + 1 \times 18 + 4 \times 9 + 6 \times 3 - 15\tfrac{3}{4} \times 15$$
$$= -122\tfrac{1}{4} \text{ tons-ft.}$$
$$B.M._{m} = 2 \times 23 + 1 \times 20 + 4 \times 11 + 8 \times 4 - 15\tfrac{3}{4} \times 17$$
$$= -125\tfrac{3}{4} \text{ tons-ft.}$$
$$B.M._{n} = 2 \times 25 + 1 \times 22 + 4 \times 13 + 10 \times 5 - 15\tfrac{3}{4} \times 19$$
$$= -125\tfrac{1}{4} \text{ tons-ft.}$$
$$B.M._{f} = 2 \times 26 + 1 \times 23 + 4 \times 14 + 11 \times 5\tfrac{1}{2} - 15\tfrac{3}{4} \times 20$$
$$= -123\tfrac{1}{2} \text{ tons-ft.}$$
$$B.M._{g} = 2 \times 27 + 1 \times 24 + 4 \times 15 + 12 \times 6 + 6 \times 1 - 15\tfrac{3}{4} \times 21$$
$$= -114\tfrac{3}{4} \text{ tons-ft.}$$
$$B.M._{h} = 2 \times 42 + 1 \times 39 + 4 \times 30 + 12 \times 21 + 6 \times 16 - 15\tfrac{3}{4} \times 36$$
$$= +24 \text{ tons-ft.}$$
$$B.M._{i} = 2 \times 50 + 1 \times 47 + 4 \times 38 + 12 \times 29 + 6 \times 24 - 15\tfrac{3}{4} \times 44 - 12\tfrac{1}{4} \times 8$$
$$= 0.$$

In order to revise the mathematical method of dealing with continuous loads on a beam it will be advisable to solve a problem.

Example 2.—A horizontal beam AB of 20 ft. span is firmly built into a wall at B and rests on a supporting column at A. It carries a distributed load of total magnitude W tons, and the bending moment at any section distant x ft. from the support at A is given by the equation

$$M = \tfrac{1}{300}(375 - 15x - x^2)x \text{ tons-ft.}$$

Deduce a corresponding equation for the shear force; determine the nature and magnitude of the load W and the pressure on the supporting column at A. Find the section of maximum and zero bending moment and its value.

$$M = \tfrac{1}{300}(375 - 15x - x^2)x.$$

By differentiation an expression for the shearing force will be obtained.

Thus $$\text{S.F.} = \tfrac{1}{300}(375 - 30x - 3x^2),$$

and by further differentiation the load expression

$$\text{load} = \tfrac{1}{300}(-30 - 6x)$$
$$= -\tfrac{1}{10} - \tfrac{1}{50}x = -(\tfrac{1}{10} + \tfrac{1}{50}x).$$

The load is a distributed load which varies from $\frac{1}{10}$ ton per foot at A to $\frac{1}{10} + \frac{1}{50} \times 20 = 0{\cdot}5$ ton per foot at B. The pressure on the support at A will be equal to the shearing force at A, that is, when $x = 0$.

Pressure on A $= \frac{1}{300}(375) = 1\frac{1}{4}$ tons.

The maximum bending moment will occur where the shearing force is zero, since the shearing-force expression is obtained by differentiating the bending-moment expression, and is in consequence an equation which gives the slope of the bending-moment curve.

Thus $$\tfrac{1}{300}(375 - 30x - 3x^2) = 0,$$

or $$3x^2 + 30x - 375 = 0,$$

i.e. $$x^2 + 10x - 125 = 0.$$

$$\therefore\ x = \frac{-10 \pm \sqrt{(100 + 500)}}{2}$$

$$= \frac{-10 \pm 24{\cdot}5}{2}$$

$$= +7{\cdot}25 \text{ ft. (negative result being impossible).}$$

$$\text{Maximum bending moment} = \tfrac{1}{300}(375 - 15 \times 7{\cdot}25 - 7{\cdot}25^2)7{\cdot}25$$

$$= \frac{214 \times 7{\cdot}25}{300}$$

$$= 5{\cdot}17 \text{ tons-ft.}$$

Zero bending moment occurs at $x = 0$ and when $x^2 + 15x - 375 = 0$,

i.e. when
$$x = \frac{-15 \pm \sqrt{(225 + 4 \times 375)}}{2}$$
$$= 13{\cdot}3 \text{ ft.}$$

Example 3.—The load on the wing of an aeroplane varies according to the equation
$$y = w_0 \frac{\sqrt{(L^2 - x^2)}}{L},$$

where w_0 is the load per unit length at the root, L is the length of the wing and x is the distance of the section XX from the root. Find expressions for the shearing force and bending moment, and the value of the maximum bending moment.

$$\text{Shearing force} = \int y\,dx = \int \frac{w_0}{L} \sqrt{(L^2 - x^2)}\,dx.$$

Let $L \sin\theta = x$, then $L\cos\theta\,d\theta = dx$.

$$\text{Shearing force} = \int \frac{w_0}{L} \sqrt{(L^2 - L^2 \sin^2\theta)}\,.\,L\cos\theta\,d\theta$$
$$= \int \frac{w_0}{L}\,.\,L^2 \cos\theta\cos\theta\,d\theta$$
$$= \int \tfrac{1}{2} w_0 L(1 + \cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2} w_0 L(\theta + \tfrac{1}{2}\sin 2\theta) + A.$$

When $x = L$ the shearing force is zero, and for x to be equal to L, θ must be equal to $\frac{1}{2}\pi$; then

$$\text{shearing force} = 0 = \tfrac{1}{2} w_0 L(\tfrac{1}{2}\pi + 0) + A,$$
$$\therefore\ A = -\tfrac{1}{4} w_0 L\pi.$$
$$\therefore\ \text{Shearing force} = \tfrac{1}{2} w_0 L(\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{1}{2}\pi).$$

$$\text{The bending moment} = \int (\text{S.F.})\,dx$$
$$= \int \tfrac{1}{2} w_0 L(\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{1}{2}\pi) L\cos\theta\,d\theta$$
$$= \int \tfrac{1}{2} w_0 L^2(\theta\cos\theta + \tfrac{1}{2}\sin 2\theta\cos\theta - \tfrac{1}{2}\pi\cos\theta)\,d\theta$$
$$= \tfrac{1}{2} w_0 L^2(\theta\sin\theta + \cos\theta - \tfrac{1}{3}\cos^3\theta - \tfrac{1}{2}\pi\sin\theta + B).$$

Known conditions: the bending moment is zero when $x = L$, i.e. $\theta = \frac{1}{2}\pi$.

$$\text{Bending moment} = 0 = \tfrac{1}{2} w_0 (L^2(\tfrac{1}{2}\pi + 0 - 0 - \tfrac{1}{2}\pi + B),$$
$$\therefore\ B = 0.$$
$$\therefore\ \text{Bending moment} = \tfrac{1}{2} w_0 L^2(\theta\sin\theta + \cos\theta - \tfrac{1}{3}\cos^3\theta - \tfrac{1}{2}\pi\sin\theta).$$

Maximum bending moment occurs at the root, i.e. when $x = 0$. But when $x = 0$, $\theta = 0$, hence

$$\text{maximum bending moment} = \tfrac{1}{2}w_0L^2(0 + 1 - \tfrac{1}{3} - 0)$$
$$= \tfrac{1}{3}w_0L^2.$$

2. Relation between Deflection, Slope, Bending Moment, Shearing Force, and Distributed Load.

It is now useful to extend the work to slope and deflection problems, for in this way we are able to obtain full solutions for varying end-conditions. In *Materials and Structures* (p. 91) it has been shown that for distributed loads there is a set of equations covering load, shearing force, bending moment, slope and deflection.

$$EI\frac{d^4y}{dx^4} = \text{load.}$$

$$EI\frac{d^3y}{dx^3} = \text{shearing force.}$$

$$EI\frac{d^2y}{dx^2} = \text{bending moment.}$$

$$\frac{dy}{dx} = \int\frac{(\text{bending moment})}{EI}\,dx$$
$$= \text{slope.}$$

$$y = \int(\text{slope})\,dx$$
$$= \text{deflection.}$$

Example 4.—A beam freely supported over a span of 20 ft. carries a load which varies uniformly from an intensity of 1 ton per foot at one

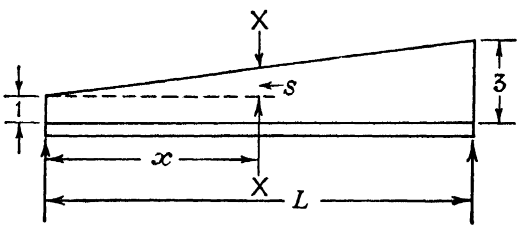


Fig. 2.2

end to 3 tons per foot at the other. If the second moment of the area of the beam section is 300 in.^{4} units, and the modulus of elasticity is 30×10^6 lb. per sq. in., calculate the deflection at the centre of the span.

For section XX,

$$EI\frac{d^4y}{dx^4} = (s+1)$$

$$= \left(\frac{2x}{L} + 1\right).$$

$$EI\frac{d^3y}{dx^3} = \frac{2x^2}{2L} + x + A.$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{3}\frac{x^3}{L} + \frac{1}{2}x^2 + Ax + B.$$

The bending moment $= 0$ when $x = 0$ and L.

Substituting $x = 0$, $B = 0$.

Substituting $x = L$,

$$EI\frac{d^2y}{dx^2} = 0 = \frac{1}{3}\frac{L^3}{L} + \frac{1}{2}L^2 + AL,$$

$$\therefore A = -\tfrac{5}{6}L.$$

$$\therefore EI\frac{d^2y}{dx^2} = \frac{1}{3}\frac{x^3}{L} + \frac{1}{2}x^2 - \frac{5}{6}Lx.$$

$$\therefore EI\frac{dy}{dx} = \frac{1}{12}\frac{x^4}{L} + \frac{1}{6}x^3 - \frac{5}{12}Lx^2 + C.$$

$$\therefore EIy = \frac{1}{60}\frac{x^5}{L} + \frac{1}{24}x^4 - \frac{5}{36}Lx^3 + Cx + D.$$

Known conditions: $y = 0$ when $x = 0$ and L.

Substituting $x = 0$, we have $D = 0$.

Substituting $x = L$,

$$EIy = 0 = \frac{1}{60}\frac{L^5}{L} + \frac{1}{24}L^4 - \frac{5}{36}L^4 + CL.$$

$$= L^4\left(\frac{6 + 15 - 50}{360}\right) + CL.$$

$$\therefore C = \tfrac{29}{360}L^3.$$

$$\therefore EIy = \frac{1}{60}\frac{x^5}{L} + \frac{1}{24}x^4 - \frac{5}{36}Lx^3 + \frac{29}{360}L^3x.$$

When $x = \frac{1}{2}L$,

$$EIy = \frac{L^5}{60 \times 32L} + \frac{L^4}{24 \times 16} - \frac{5L^4}{36 \times 8} + \frac{29L^4}{360 \times 2}$$

$$= \frac{150L^4}{5760}.$$

$$\therefore y = \frac{150L^4 \times 2240}{5760 \times 30 \times 10^6 \times 300}$$

$$= \frac{L \times (12L)^3 \times 150 \times 2240}{5760 \times 30 \times 10^6 \times 300},\text{ where } L \text{ is in feet.}$$

Bearing in mind that the loading was given in tons per foot,

$$y = \frac{20 \times (240)^3 \times 150 \times 2240}{5760 \times 30 \times 10^6 \times 300}$$

$$= 1{\cdot}792 \text{ in.}$$

3. Propped Beams.

If a beam—cantilever or supported—is further supported by a prop, the normal method of statics will not enable the load on the prop to be calculated. It will be necessary to obtain the bending-moment expression in terms of the unknown load on the prop, integrate twice and apply the known conditions of deflection to obtain the values of the integration constants.

Example 5.—A cantilever of span L carries a uniformly distributed load w per unit length over the complete beam. If the free end is supported by a prop so that it is level with the fixed end, find the load on the prop.

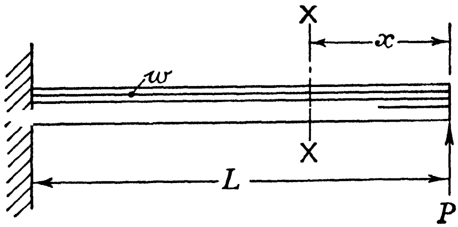


Fig. 2.3

For section XX,

$$\text{B.M.}_{XX} = -Px + \tfrac{1}{2}wx^2 = EI\frac{d^2y}{dx^2}.$$

$$\therefore\ EI\frac{dy}{dx} = \int(-Px + \tfrac{1}{2}wx^2)\,dx$$

$$= -\tfrac{1}{2}Px^2 + \tfrac{1}{6}wx^3 + A.$$

$$\therefore\ EIy = \int(-\tfrac{1}{2}Px^2 + \tfrac{1}{6}wx^3 + A)\,dx$$

$$= -\tfrac{1}{6}Px^3 + \tfrac{1}{24}wx^4 + Ax + B.$$

Known conditions: $y = 0$ when $x = 0$ and L,

$dy/dx = 0$ when $x = L$.

Substituting $x = 0$, $y = 0$, we have $B = 0$.

Substituting $x = L$, $y = 0$,

$$EIy = 0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{24}wL^4 + AL.$$

$$\therefore\ A = \tfrac{1}{6}PL^2 - \tfrac{1}{24}wL^3.$$

Substituting $x = L$, $dy/dx = 0$,

$$EI\frac{dy}{dx} = 0 = -\tfrac{1}{2}PL^2 + \tfrac{1}{6}wL^3 + \tfrac{1}{6}PL^2$$
$$= -\tfrac{1}{3}PL^2 + \tfrac{1}{8}wL^3$$
$$\therefore P = \tfrac{3}{8}wL = \tfrac{3}{8}W.$$

If in the above example the conditions stated that the deflection at the free end was k, then

$$y = k \text{ when } x = 0;$$
$$y = 0 \text{ when } x = L;$$
$$dy/dx = 0 \text{ when } x = L.$$

Substituting $y = k$ when $x = 0$,

$$EIy = EIk = B.$$

Substituting $y = 0$, $x = L$,

$$EIy = 0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{24}wL^4 + AL + EIk.$$
$$\therefore A = \tfrac{1}{6}PL^2 - \tfrac{1}{24}wL^3 - \frac{EIk}{L}.$$

Substituting $dy/dx = 0$ when $x = L$,

$$EI\frac{dy}{dx} = 0 = -\tfrac{1}{2}PL^2 + \tfrac{1}{6}wL^3 + \tfrac{1}{6}PL^2 - \tfrac{1}{24}wL^3 - \frac{EIk}{L}.$$
$$\therefore P = \tfrac{3}{8}wL - \frac{3kEI}{L^3}.$$

Generally the problem of the two ends being at different levels is due to the prop shortening under load, and also on account of its sinking. If the prop sinks or shortens a distance e per unit load carried, then the difference in level will be Pe. The procedure will then be as above with Pe replacing k, so that

$$P = \tfrac{3}{8}wL - \frac{3PeEI}{L^3}$$

or

$$P = \frac{\tfrac{3}{8}wL \,.\, L^3}{L^3 + 3eEI}.$$

Example 6.—A beam of uniform section and of length $2l$ is supported freely at its ends A and B and also at the centre C. The supports A and B are rigid, but the support at C deflects a depth which is u times the load it carries. If the beam is subjected to a total load W uniformly distributed along its length, show that the load on the centre support is

$$\frac{\tfrac{5}{8}W}{1 + 6EIu/l^3}.$$

Reaction at A = reaction at B = S.

Bending moment at XX $= Sx - \frac{1}{2}wx^2$,

i.e. $$EI\frac{d^2y}{dx^2} = -Sx + \tfrac{1}{2}wx^2.$$

$$\therefore\ EI\frac{dy}{dx} = -\tfrac{1}{2}Sx^2 + \tfrac{1}{6}wx^3 + A.$$

$$\therefore\ EIy \quad \tfrac{1}{6}Sx^3 + \tfrac{1}{24}wx^4 + Ax + B.$$

Known conditions are:

$y = 0$ when $x = 0$; . . . . (i)

$y = Pu$, where P is load on C, when $x = l$; . . (ii)

$\frac{dy}{dx} = 0$ when $x = l$. . . . . (iii)

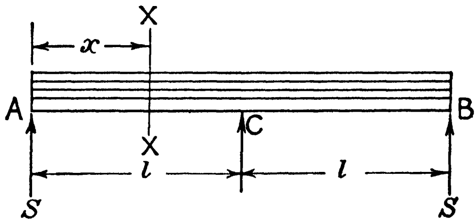


Fig 2.4

From (i), $$y = 0 = 0 + B, \quad \therefore\ B = 0.$$

From (ii), $$EIy = EIPu = -\tfrac{1}{6}Sl^3 + \tfrac{1}{24}wl^4 + Al,$$

$$\therefore\ A = \tfrac{1}{6}Sl^2 - \tfrac{1}{24}wl^3 + \frac{EIPu}{l}.$$

From (iii), $$EI\frac{dy}{dx} = 0 \quad -\tfrac{1}{2}Sl^2 + \tfrac{1}{6}wl^3 + A,$$

$$\therefore\ A \quad \tfrac{1}{2}Sl^2 - \tfrac{1}{6}wl^3.$$

But $$A \quad \tfrac{1}{6}Sl^2 - \tfrac{1}{24}wl^3 + \frac{EIPu}{l},$$

$$\therefore\ \tfrac{1}{6}Sl^2 - \tfrac{1}{24}wl^3 + \frac{EIPu}{l} - \tfrac{1}{2}Sl^2 - \tfrac{1}{6}wl^3. \quad . . . . . . \text{(iv)}$$

Now sum of loads on A, B and C must be $2wl$, and the loads on A and B from symmetry are each equal to S.

$$\therefore\ 2S + P = 2wl,$$

i.e. $$S - \tfrac{1}{2}(2wl - P).$$

Substituting in (iv),

$$\tfrac{1}{12}(2wl - P)l^2 - \tfrac{1}{24}wl^3 + \frac{EIPu}{l} = \tfrac{1}{4}(2wl - P)l^2 - \tfrac{1}{6}wl^3,$$

i.e.
$$\tfrac{1}{6}wl^3 - \tfrac{1}{12}Pl^2 - \tfrac{1}{24}wl^3 + \frac{EIPu}{l} = \tfrac{1}{2}wl^3 - \tfrac{1}{4}Pl^2 - \tfrac{1}{6}wl^3,$$

i.e.
$$\tfrac{1}{6}Pl^2 + \frac{EIPu}{l} = \tfrac{5}{24}wl^3,$$

$$\therefore\ P = \frac{\tfrac{5}{24}wl^3}{\tfrac{1}{6}l^2 + EIu/l}$$

$$= \frac{\tfrac{5}{24}wl^3}{(1 + 6EIu/l^3)\tfrac{1}{6}l^2}$$

$$= \frac{\tfrac{5}{4}wl}{(1 + 6EIu/l^3)}.$$

But $W = 2wl$,
$$\therefore\ P = \frac{\tfrac{5}{8}W}{(1 + 6EIu/l^3)}.$$

This problem could have been solved more quickly if the deflection at the centre of a simply supported beam of span $2l$ with distributed load had been found, and the central deflection produced by an upward central concentrated load.

Deflection at mid-span for uniformly distributed load

$$= \frac{5}{384}\frac{W(2l)^3}{EI}$$

$$= \frac{5}{48}\frac{Wl^3}{EI}. \qquad \text{(a)}$$

Upward deflection produced by concentrated load P at mid-span will be

$$\frac{1}{48}\frac{P(2l)^3}{EI}$$

$$= \frac{1}{6}\frac{Pl^3}{EI}. \qquad \text{(b)}$$

The deflection given by (a) equals the deflection given by (b) plus the amount the prop sinks, namely, Pu.

$$\therefore\ \frac{5}{48}\frac{Wl^3}{EI} = \frac{1}{6}\frac{Pl^3}{EI} + Pu$$

$$= \frac{Pl^3}{6EI}\left(1 + \frac{6EIu}{l^3}\right),$$

or
$$P = \frac{\tfrac{5}{8}W}{(1 + 6EIu/l^3)}.$$

4. Beams with Several Concentrated Loads.

When a beam is loaded with several concentrated loads the problem of finding slopes and deflections will become long and involved. The work may be simplified by adopting the following procedure:

(*a*) *Cantilever.*—Taking the cantilever of length a_4 with loads W_1, W_2, W_3 and W_4 applied at b, c, d and e respectively, the bending moment at section XX distance x from the fixed end is

$$M_X = W_1(a_1 - x) + W_2(a_2 - x) + W_3(a_3 - x) + W_4(a_4 - x)$$
$$= \Sigma W(a - x),$$

i.e. $$EI\frac{d^2y}{dx^2} = -\Sigma W(a - x).$$

$$\therefore\ EI\frac{dy}{dx} = \tfrac{1}{2}\Sigma W(a - x)^2 + A.$$

$$\therefore\ EIy = -\tfrac{1}{6}\Sigma W(a - x)^3 + Ax + B.$$

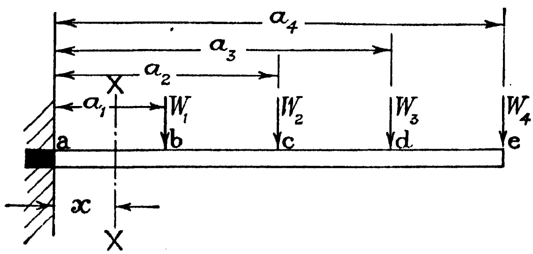


Fig. 2.5

When $x = 0$, $dy/dx = 0$ and $y = 0$.

$$\therefore\ EI\frac{dy}{dx} = 0 = \tfrac{1}{2}\Sigma Wa^2 + A,$$

$$\therefore\ A = -\tfrac{1}{2}\Sigma Wa^2.$$

$$EIy = 0 = -\tfrac{1}{6}\Sigma Wa^3 + B,$$

$$\therefore\ B = +\tfrac{1}{6}\Sigma Wa^3.$$

When the section under review is between a and b all the terms are taken into account, but when the section is between b and c the the term $W_1(a_1 - x)$ is left out. Similarly between c and d the terms $W_1(a_1 - x)$ and $W_2(a_2 - x)$ are rejected, i.e. whenever $(a - x)$ is negative the term is rejected.

Hence $$EI\frac{dy}{dx} = \tfrac{1}{2}\Sigma W(a-x)^2 - \tfrac{1}{2}\Sigma Wa^2.$$

$$\therefore\ EIy = -\tfrac{1}{6}\Sigma W(a-x)^3 - \tfrac{1}{2}x\Sigma Wa^2 + \tfrac{1}{6}\Sigma Wa^3.$$

(*b*) *Simply supported beam.*—A simply supported beam of span L is loaded at b, c, d and e with loads of W_1, W_2, W_3 and W_4 respectively.

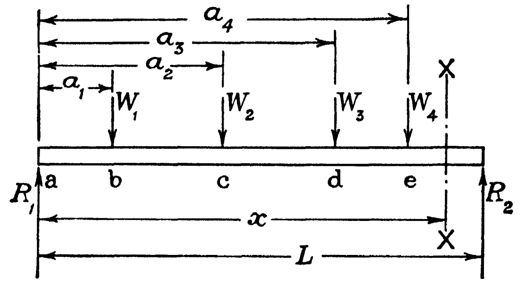


Fig. 2.6

$$\text{The reaction at a} = R_1 = \frac{W_1(L-a_1) + W_2(L-a_2) + \ldots}{L}$$

$$= \frac{\Sigma W(L-a)}{L}.$$

Taking section XX between e and R_2, the bending moment expression is

$$M_x = R_1x - W_1(x-a_1) - W_2(x-a_2) + \ldots$$
$$= R_1x - \Sigma W(x-a),$$

i.e. $$EI\frac{d^2y}{dx^2} = R_1x - \Sigma W(x-a),$$

$$\therefore\ EI\frac{dy}{dx} = \tfrac{1}{2}R_1x^2 - \tfrac{1}{2}\Sigma W(x-a)^2 + A.$$

$$\therefore\ EIy = \tfrac{1}{6}R_1x^3 - \tfrac{1}{6}\Sigma W(x-a)^3 + Ax + B.$$

Whenever the expression $(x - a)$ becomes negative the term may be neglected. Known conditions:

$$y = 0 \text{ when } x = 0,\ \therefore\ B = 0;$$
$$y = 0 \text{ when } x = L.$$

$$\therefore\ EIy = 0 = \tfrac{1}{6}R_1L^3 - \tfrac{1}{6}\Sigma W(L-a)^3 + AL.$$

$$\therefore\ A = \frac{1}{6L}\{\Sigma W(L-a)^3 - L^2\Sigma W(L-a)\}.$$

Hence $$EI\frac{dy}{dx} = \frac{1}{2L}\Sigma W(L-a)x^2 - \tfrac{1}{2}\Sigma W(x-a)^2$$
$$+\frac{1}{6L}\{\Sigma W(L-a)^3 - L^2\Sigma W(L-a)\}.$$

$$\therefore\ EIy = \frac{1}{6L}\Sigma W(L-a)x^3 - \tfrac{1}{6}\Sigma W(x-a)^3$$
$$+\frac{1}{6L}\{\Sigma W(L-a)^3x - L^2\Sigma W(L-a)x\}.$$

(c) *Distributed load on portion of cantilever.*—When a distributed load is placed on a beam, but only a portion of the span is covered, the problem may be solved by assuming that the load is continued to the section under review. In order that no change be made in the problem an equal but opposite force is also applied over the portion of the beam over which the load is continued.

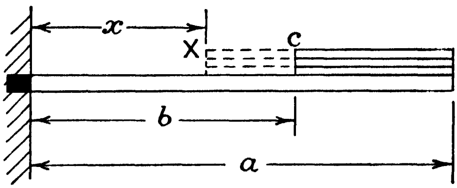


Fig. 2.7

A cantilever of length a has a load of w per unit length over a length of beam $(a-b)$, the load starting at the free end. For section XX imagine the load continued from C to X and an equal but opposite force from X to C.

Bending moment at XX $= \frac{1}{2}w(a-x)^2 - \frac{1}{2}w(b-x)^2$,

i.e. $$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}w(a-x)^2 + \tfrac{1}{2}w(b-x)^2.$$

$$\therefore\ EI\frac{dy}{dx} = \tfrac{1}{6}w(a-x)^3 - \tfrac{1}{6}w(b-x)^3 + A,$$

and $$EIy = -\tfrac{1}{24}w(a-x)^4 + \tfrac{1}{24}w(b-x)^4 + Ax + B.$$

Known conditions: $dy/dx = 0$ and $y = 0$ when $x = 0$,

$$\therefore\ A = -\tfrac{1}{6}w(a^3 - b^3),$$

and $$B = \tfrac{1}{24}w(a^4 - b^4).$$

Then

$$EI\frac{dy}{dx} = \tfrac{1}{6}w(a-x)^3 - \tfrac{1}{6}w(b-x)^3 - \tfrac{1}{6}w(a^3-b^3),$$

and

$$EIy = -\tfrac{1}{24}w(a-x)^4 + \tfrac{1}{24}w(b-x)^4 - \tfrac{1}{6}w(a^3-b^3)x + \tfrac{1}{24}w(a^4-b^4).$$

(*d*) *Distributed load on portion of simply supported beam.*—When a simply supported beam carries a distributed load over part of the span, a similar procedure may be adopted.

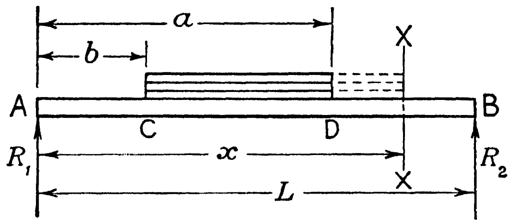


Fig. 2.8

A beam AB is simply supported at A and B and carries a uniformly distributed load of w units per unit length over the portion CD, C being a distance b from A and D being at a distance a.

$$R_1 = \frac{w(a-b)}{2L}(2L-a-b).$$

For section XX imagine the load is continued from D to XX, and an equal but opposite force acts over the same portion; then the bending moment at XX is

$$M_X = R_1x - \tfrac{1}{2}w(x-b)^2 + \tfrac{1}{2}w(x-a)^2,$$

i.e. $$EI\frac{d^2y}{dx^2} = R_1x - \tfrac{1}{2}w(x-b)^2 + \tfrac{1}{2}w(x-a)^2.$$

$$\therefore\ EI\frac{dy}{dx} = \tfrac{1}{2}R_1x^2 - \tfrac{1}{6}w(x-b)^3 + \tfrac{1}{6}w(x-a)^3 + A.$$

$$\therefore\ EIy = \tfrac{1}{6}R_1x^3 - \tfrac{1}{24}w(x-b)^4 + \tfrac{1}{24}w(x-a)^4 + Ax + B.$$

Known conditions are $y = 0$ when $x = 0$ and L.

When $x = 0$, $y = 0$, $\therefore\ B = 0$,

neglecting the terms when $(x-b)$ becomes negative.

When $x = L$,

$$EIy = 0 = \tfrac{1}{6}R_1L^3 - \tfrac{1}{24}w(L-b)^4 + \tfrac{1}{24}w(L-a)^4 + AL.$$

$$\therefore\ A = -\tfrac{1}{6}R_1L^2 + \tfrac{1}{24}\frac{w}{L}(L-b)^4 - \frac{w}{24L}(L-a)^4.$$

Thus
$$EI\frac{dy}{dx} = \frac{w}{2L}(a-b)(2L-a-b)\frac{x^2}{2} - \tfrac{1}{6}w(x-b)^3 + \tfrac{1}{6}w(x-a)^3 - \frac{w}{2L}(a-b)(2L-a-b)\frac{L^2}{6} + \tfrac{1}{24}\frac{w}{L}(L-b)^4 - \frac{w}{24L}(L-a)^4,$$

and
$$EIy = \frac{w}{2L}(a-b)(2L-a-b)\frac{x^3}{6} - \tfrac{1}{24}w(x-b)^4 + \tfrac{1}{24}w(x-a)^4 - \frac{w}{2L}(a-b)(2L-a-b)\frac{L^2x}{6} + \tfrac{1}{24}\frac{w}{L}(L-b)^4x - \frac{w}{24L}(L-a)^4x.$$

5. Beams with Varying Cross-sections.

When the cross-section of the beam changes there is a change in the second moment of the area which must be taken into account. If the change is gradual and can be expressed algebraically, then the variation in the second moment can be incorporated in the integration; but if the change is sudden and cannot be expressed mathematically it will be necessary to solve the problem in separate parts. The following examples will show how the problems might be tackled.

Example 7.—A cantilever, 8 ft. span of rectangular section, is 8 in. broad throughout, but the depth tapers uniformly from 10 in. at the

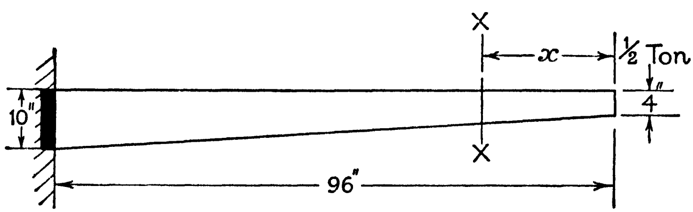


Fig. 2.9

wall to 4 in. at the free end. A load of $\frac{1}{2}$ ton is carried at the free end. Determine expressions for slope and deflection and the value of the deflection at the free end. $E = 1{\cdot}5 \times 10^6$ lb. per sq. in.

$$I_{xx} = \tfrac{1}{12} \times 8d^3$$
$$= \tfrac{1}{12} \times 8\left(4 + \frac{6x}{96}\right)^3$$
$$= \tfrac{2}{3}(4 + \tfrac{1}{16}x)^3$$
$$= \frac{2}{3 \times 16^3}(64 + x)^3.$$

$$E\frac{d^2y}{dx^2} = \frac{-\tfrac{1}{2}x}{\dfrac{2}{3 \times 16^3}(64 + x)^3},$$

$$\therefore \frac{4E}{3 \times 16^3}\frac{d^2y}{dx^2} = \frac{-x}{(64 + x)^3}.$$

Let $u = 64 + x$, then $x = u - 64$ and $dx = du$.

Substituting, $$\frac{4E}{3 \times 16^3}\frac{d^2y}{du^2} = \frac{-(u - 64)}{u^3}$$
$$= -\frac{1}{u^2} + \frac{64}{u^3},$$

$$\therefore \frac{4E}{3 \times 16^3}\frac{dy}{du} = \frac{1}{u} - \frac{32}{u^2} + A,$$

and $$\frac{4E}{3 \times 16^3}y = \log_e u + \frac{32}{u} + Au + B.$$

Now y and $\frac{dy}{dx}$ or $\frac{dy}{du} = 0$ when $u = (64 + 96) = 160$.

$$\therefore \frac{dy}{dx} = 0 = \frac{1}{160} - \frac{32}{160^2} + A,$$
$$\therefore A = -\tfrac{1}{200}.$$
$$y = 0 = \log_e 160 + \tfrac{32}{160} - \tfrac{160}{200} + B$$
$$= 5{\cdot}0752 + {\cdot}2 - {\cdot}8 + B,$$
$$\therefore B = -4{\cdot}4752.$$

$$\frac{dy}{dx} = \frac{3 \times 16^3 \times 2240}{4 \times 1{\cdot}5 \times 10^6}\left[\frac{1}{x + 64} - \frac{32}{(x + 64)^2} - \frac{1}{200}\right]$$
$$= 4{\cdot}587\left[\frac{1}{x + 64} - \frac{32}{(x + 64)^2} - \frac{1}{200}\right],$$

and $$y = 4{\cdot}587\left[\log_e(x + 64) + \frac{32}{x + 64} - \frac{x + 64}{200} - 4{\cdot}4752\right].$$

When $x = 0$, $y = 4{\cdot}587\,[\log_e 64 + \tfrac{32}{64} - \tfrac{64}{200} - 4{\cdot}4752]$
$= 4{\cdot}587\,[4{\cdot}1589 + {\cdot}5 - {\cdot}32 - 4{\cdot}4752]$
$= -4{\cdot}587 \times {\cdot}1363$
$= -{\cdot}6252$ in.

Example 8.—A horizontal shaft, 9 ft. long between its bearings, is 4 in. in diameter at the middle third of its length, and 3 in. in diameter at the end thirds. It carries a concentrated load of 1 ton at the middle of its length. Take the shaft as simply supported at its bearings, neglect its weight and take $E = 13{,}000$ tons per sq. in. Find the deflection under the load.

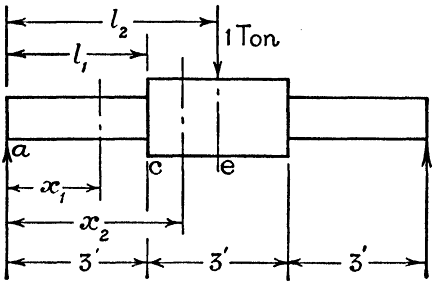


Fig. 2.10

$$I_1 = \frac{\pi}{64}3^4 = 4 \text{ in.}^4 \text{ units.}$$

$$I_2 = \frac{\pi}{64}4^4 = 12{\cdot}6 \text{ in.}^4 \text{ units.}$$

For section ac, $$EI_1\frac{d^2y}{dx^2} = \tfrac{1}{2}x_1.$$

$$EI_1\frac{dy}{dx} = \tfrac{1}{4}x_1^2 + A_1.$$

$$EI_1y = \tfrac{1}{12}x_1^3 + A_1x_1 + B_1.$$

Known conditions: $y = 0$ when $x = 0$, $\therefore B_1 = 0$.

Substituting the value of I_1,

$$E\frac{dy}{dx} = \tfrac{1}{16}x_1^2 + \tfrac{1}{4}A_1, \quad \text{. . . . . (i)}$$

and $$Ey = \tfrac{1}{48}x_1^3 + \tfrac{1}{4}A_1x_1. \quad \text{. . . . . . . (ii)}$$

For section ce, $$EI_2\frac{d^2y}{dx^2} = \tfrac{1}{2}x_2.$$

$$EI_2\frac{dy}{dx} = \tfrac{1}{4}x_2^2 + A_2.$$

$$EI_2y = \tfrac{1}{12}x_2^3 + A_2x_2 + B_2.$$

Known conditions: $\frac{dy}{dx} = 0$ when $x = l_2$.

$$\therefore EI_2 \frac{dy}{dx} = 0 = \tfrac{1}{4}l_2^2,$$

$$\therefore A_2 = -\tfrac{1}{4}l_2^2 = -\tfrac{1}{4} \times 54^2 = -729.$$

Substituting the value of I_2,

$$E\frac{dy}{dx} = \frac{x_2^2}{50 \cdot 4} - \frac{729}{12 \cdot 6}$$

$$= \frac{x_2^2}{50 \cdot 4} - 58, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(iii)}$$

and

$$Ey = \frac{x_2^3}{151 \cdot 2} - 58\, x_2 + B_2. \quad . \ . \ . \ . \ . \ . \quad \text{(iv)}$$

In order to find the values of the constants A_1 and B_2 it will be necessary to equate the slope given by equations (i) and (iii) and the deflections given by equations (ii) and (iv) for position c, i.e. $x_1 = x_2 = l_1 = 36$ in.

$$E\frac{dy}{dx} = \frac{l_1^2}{16} + \frac{A_1}{4} = \frac{l_1^2}{50 \cdot 4} - 58,$$

$$\therefore \frac{A_1}{4} = l_1^2 \frac{16 - 50 \cdot 4}{16 \times 50 \cdot 4} - 58$$

$$= \frac{-34 \cdot 4 \times 36^2}{16 \times 50 \cdot 4} - 58$$

$$-55 \cdot 3 - 58$$

$$= -113 \cdot 3,$$

or

$$A_1 \quad -453 \cdot 2.$$

$$Ey = \frac{l_1^3}{48} - \frac{453 \cdot 2 l_1}{4} = \frac{l_1^3}{151 \cdot 2} - 58 l_1 + B_2,$$

i.e.

$$l_1^3 \frac{151 \cdot 2 - 48}{48 \times 151 \cdot 2} - 55 \cdot 3 l_1 = B_2,$$

i.e.

$$663 \cdot 5 - 1990 \cdot 8 = B_2,$$

$$\therefore B_2 = -1327 \cdot 3.$$

For deflection under the load, equation (iv) must be used.

$$y = \frac{1}{E}\left[\frac{l_2^3}{151 \cdot 2} - 58 l_2 - 1327 \cdot 3\right]$$

$$= \frac{1}{13{,}000}\left[\frac{54^3}{151 \cdot 2} - 58 \times 54 - 1327 \cdot 3\right]$$

$$= \frac{1}{13{,}000}(1041 \cdot 5 - 3132 - 1327 \cdot 3)$$

$$= -0 \cdot 263 \text{ in.}$$

6. Deflections from Bending-moment Diagram.

It is sometimes easier to ascertain the slope and deflection of a beam by the following method rather than by mathematical treatment.

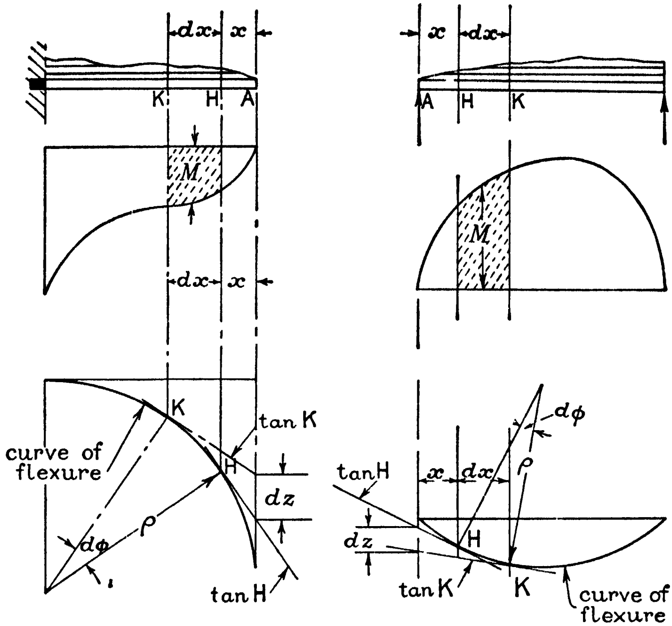


Fig. 2.11

The length of the arc HK $= \rho \,.\, d\phi$, where ρ is the radius of curvature and $d\phi$ the angle subtended at the centre of curvature. The arc HK is approximately equal to the horizontal distance dx when the curvature is small, therefore $dx = \rho \,.\, d\phi$. But from the stress in beams

$$M/I = E/\rho,$$

$$\therefore\ dx = \frac{EI}{M}\,d\phi,$$

or
$$d\phi = \frac{M\,dx}{EI}.$$

But $M\,dx =$ area of element of bending-moment diagram,

$$\therefore\ d\phi = \frac{dA}{EI},$$

or
$$\phi = \int_0^L \frac{M\,dx}{EI} = \frac{A}{EI},$$

where A is the area of the bending-moment diagram between the positions where $x = 0$ and L. ϕ will be the slope of the tangent at the datum line ($x = 0$) if the tangent to the point at distance L from the datum line is horizontal.

The vertical distance on the datum line passing through A between the tangents at H and K is dz, so that

$$dz = x\,d\phi \text{ approximately (as the curvature is small)}$$

$$= x\frac{M\,dx}{EI}$$

$$= \frac{Mx\,dx}{EI}.$$

$$\therefore\ z = \int_0^L \frac{Mx\,dx}{EI}$$

$$= \frac{A\bar{x}}{EI},$$

where $\bar{x}$ is the distance of centre of area from datum line.

The distance z will be the vertical deflection when the tangent at distance L from the datum line is horizontal. It will therefore be possible to use the above methods to find both the maximum slope and maximum deflection, provided the positions of zero slope and maximum deflection are known.

Example 9.—A cantilever of length L has a concentrated load W at the free end; find expressions for the maximum slope and deflection.

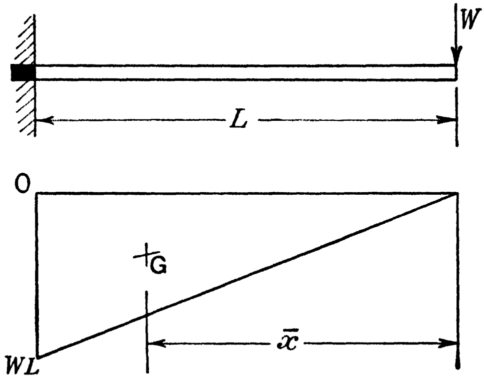


Fig. 2.12

The bending-moment diagram will be as shown above.

Area of bending-moment diagram $= \frac{1}{2}WL \times L,$

i.e. $$A = \tfrac{1}{2}WL^2.$$

The distance of centre of area from free end $= \bar{x} = \frac{2}{3}L.$

The maximum slope $$= \phi = \frac{A}{EI}$$
$$= \frac{1}{2}\frac{WL^2}{EI}.$$

The maximum deflection $$= z = \frac{A\bar{x}}{EI}$$
$$= \frac{\frac{1}{2}WL^2 \cdot \frac{2}{3}L}{EI}$$
$$= \frac{1}{3}\frac{WL^3}{EI}.$$

Example 10.—A simply supported beam carries a uniformly distributed load w over a span L; find expressions for the maximum slope and deflection.

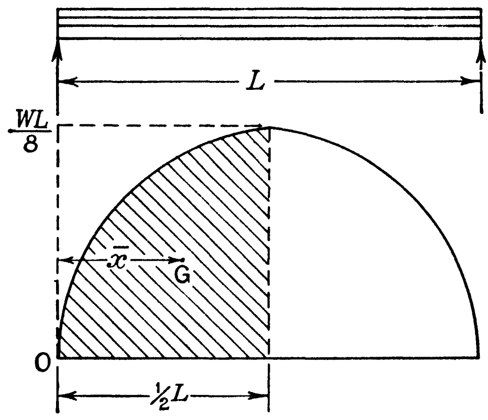


Fig. 2.13

Area of bending-moment diagram between datum line and point of maximum deflection
$$= \tfrac{2}{3} \times \tfrac{1}{8}WL \times \tfrac{1}{2}L$$
$$= \tfrac{1}{24}WL^2.$$

Distance of centre of area from datum line
$$= \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}.$$

The maximum slope $= \phi = \dfrac{A}{EI}$

$$= \frac{\frac{1}{24}WL^2}{EI}$$

$$= \frac{WL^2}{24EI}.$$

The maximum deflection $= z = \dfrac{A\bar{x}}{EI}$

$$= \frac{\frac{1}{24}WL^2 \times \frac{5}{16}L}{EI}$$

$$= \frac{5}{384}\frac{WL^3}{EI}.$$

Example 11.—A plate girder, 40 ft. span, carries 2 loads of 8 tons each at points 10 ft. on either side of mid-span. The second moment of the area is 1440 in.[4] units and the modulus of elasticity of the material is 13,000 tons per sq. in.; find the central deflection of the girder.

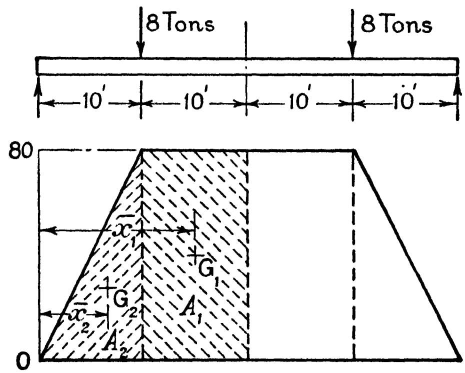


Fig. 2.14

$$A\bar{x} = A_1\bar{x}_1 + A_2\bar{x}_2$$

$$= 10 \times 80 \times 15 + \tfrac{1}{2}(10 \times 80)\tfrac{20}{3}$$

$$= 12{,}000 + \tfrac{1}{3} \times 8000$$

$$= \tfrac{1}{3} \times 44{,}000 \text{ in feet and tons.}$$

$$z = \frac{A\bar{x}}{EI}$$

$$= \frac{\frac{1}{3} \times 44{,}000 \times 1728}{13{,}000 \times 1440}$$

$$= 1{\cdot}36 \text{ in.}$$

It has been shown that the maximum deflection may be obtained by integrating $(Mx/EI)\,dx$; by using this method the solution of some problems is considerably simplified. The following example has already been solved by the full mathematical method (see p. 41), therefore by solving with $\int (Mx/EI)\,dx$ a comparison will be obtained.

Example 12.—A horizontal shaft 9 ft. long between its bearings is 4 in. in diameter at the middle third of its length and 3 in. in diameter at the end thirds. It carries a load of 1 ton at the middle of its length. Take the shaft as simply supported at its bearings, neglect its weight and take $E = 13{,}000$ tons per sq. in. Find the deflection under the load.

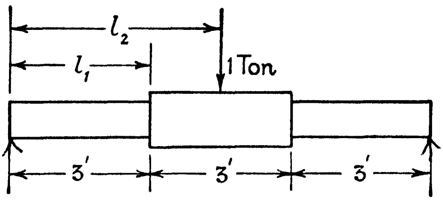


Fig. 2.15

The bending moment at distance x is $\frac{1}{2}Wx$. As there is a change in diameter of the shaft the deflection expression will consist of two parts.

$$z = \int_0^{l_1} \frac{Mx\,dx}{EI_1} + \int_{l_1}^{l_2} \frac{Mx\,dx}{EI_2}$$

$$= \int_0^{l_1} \frac{1}{2}\frac{Wx\,.\,x\,dx}{EI_1} + \int_{l_1}^{l_2} \frac{1}{2}\frac{Wx\,.\,x\,dx}{EI_2}$$

$$= \frac{1}{2}\frac{W}{E}\left[\int_0^{l_1} \frac{x^2\,dx}{I_1} + \int_{l_1}^{l_2} \frac{x^2\,dx}{I_2}\right]$$

$$= \frac{1}{2}\frac{W}{E}\left[\frac{l_1^3}{3I_1} + \frac{l_2^3 - l_1^3}{3I_2}\right]$$

$$= \frac{W \times 64}{3 \times 2 \times E\pi}\left[\frac{l_1^3}{d_1^4} + \frac{l_2^3 - l_1^3}{d_2^4}\right]$$

$$= \frac{1 \times 64 \times 1728}{6\pi \times 13{,}000}\left[\frac{3^3}{3^4} + \frac{4{\cdot}5^3 - 3^3}{4^4}\right].$$

$$= 0{\cdot}263 \text{ in.}$$

This is the result obtained by the mathematical method of finding the deflection—see Example 8.

7. Deflection of Reinforced-concrete Beams.

The deflection of a beam due to bending may be stated in the form $kWL^3/(EI)$, where k is a constant depending on the form of loading and the manner in which the beam is supported. For a composite beam, such as a reinforced-concrete beam, it is necessary to replace the product EI by (E_c × second moment of area of concrete under compression) + (E_s × second moment of area of steel), both second moments being taken about the neutral surface of the beam. The area of concrete in tension is neglected.

(*a*) *Rectangular beam of reinforced concrete.*

$I_c = \frac{1}{3}B(kD)^3$, since the concrete takes only compression.

$I_s = a_s(D - kD)^2$, neglecting second moment about its own axis.

$$\therefore\ EI = E_c \cdot \tfrac{1}{3}B(kD)^3 + E_s a_s(D - kD)^2.$$

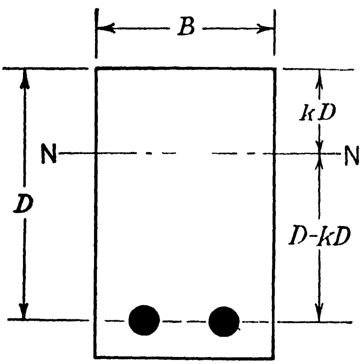


Fig. 2.16

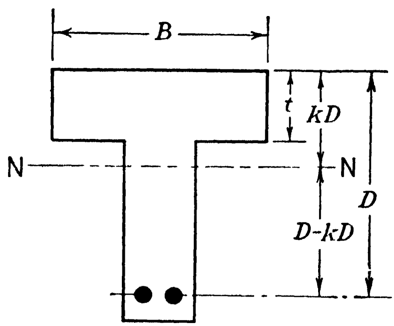


Fig. 2.17

(*b*) *Tee reinforced-concrete beam.*

I_c = second moment about own axis + area × (distance between own axis and neutral surface)^2

$= \frac{1}{12}Bt^3 + Bt(kD - \frac{1}{2}t)^2$, neglecting concrete in compression in leg of tee.

$I_s = a_s(D - kD)^2$, neglecting second moment about its own axis.

Hence $\quad EI = E_c[\frac{1}{12}Bt^3 + Bt(kD - \frac{1}{2}t)^2] + E_s a_s(D - kD)^2.$

Example 13.—A rectangular beam is simply supported over a span of 20 ft. The beam is 12 in. by 19 in., and is reinforced by five $\frac{5}{8}$-in. bars

placed 17 in. below the surface. What deflection at the centre of the span would be produced by a uniformly distributed load of 500 lb. per ft. run? $E_c = 2 \times 10^6$ lb. per sq. in.; $E_s = 30 \times 10^6$ lb. per sq. in.

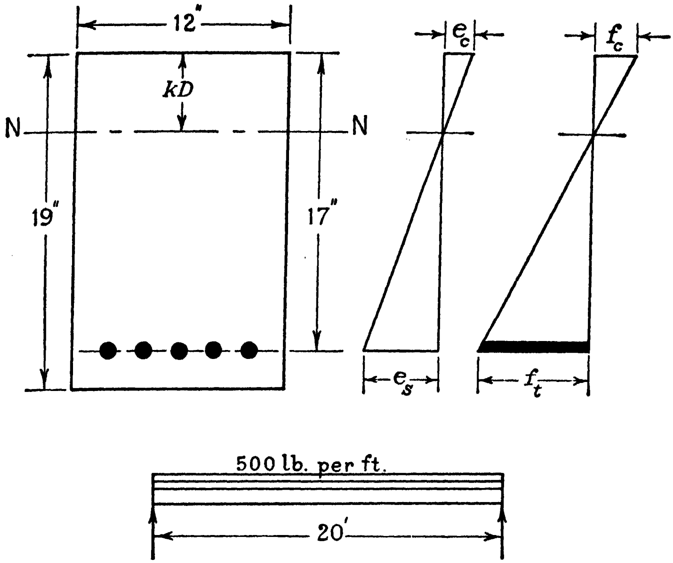


Fig. 2.18

From equations (3) and (4) on pp. 51 and 52 of *Materials and Structures*,

$$\frac{f_c}{f_t} = \frac{E_c k}{E_s(1 - k)} = \frac{2a_s}{kD \times B}$$

$$= \frac{2r}{k}, \text{ where } r = \frac{a_s}{BD}.$$

Let $$\frac{E_s}{E_c} = n,$$

then $$\frac{k}{n(1 - k)} = \frac{2r}{k},$$

or $$k = -rn \pm \sqrt{(r^2 n^2 + 2rn)}.$$

Area of bars $= a_s = 5 \times \frac{1}{4}\pi \times (\frac{5}{8})^2 = 1{\cdot}535$ sq. in.

$$r = \frac{a_s}{BD} = \frac{1{\cdot}535}{17 \times 12} = {\cdot}0075.$$

$$k = -{\cdot}0075 \times 15 \pm \sqrt{[({\cdot}0075 \times 15)^2 + 2 \times {\cdot}0075 \times 15]}$$

$$= 0{\cdot}374 \quad (k \text{ must be the positive}).$$

$$I_c = \tfrac{1}{3}B(kD)^3 = \tfrac{1}{3} \times 12(\cdot 374 \times 17)^3$$
$$= 1028 \text{ in.}^4 \text{ units.}$$
$$I_s = a_s D^2(1 - k)^2$$
$$= 1\cdot 535 \times 17^2(1 - \cdot 374)^2$$
$$= 174 \text{ in.}^4 \text{ units.}$$
$$EI = E_c I_c + E_s I_s$$
$$= 2 \times 10^6 \times 1028 + 30 \times 10^6 \times 174$$
$$= 2 \times 10^6[1028 + 15 \times 174]$$
$$= 2 \times 10^6 \times 3638 \text{ lb. in.}^2$$
$$\text{Deflection} = \frac{5}{384}\frac{WL^3}{EI}$$
$$= \frac{5 \times 500 \times 20 \times 20^3 \times 12^3}{384 \times 2 \times 10^6 \times 3638}$$
$$= 0\cdot 247 \text{ in.}$$

EXAMPLES

1. A rolled-steel joist supported at the ends over a span of 20 ft. carries a load which varies uniformly from $1\frac{1}{2}$ tons per foot run over one support to $2\frac{1}{2}$ tons per foot run over the other. Find the maximum bending moment on the beam due to this load. [100·4 tons-ft.]

2. A girder 24 ft. span carries a brick wall (weight 1 cwt. per c. ft.) over an opening. The amount of brickwork supported by the girder may be assumed to be a triangular prism 18 in. thick, 24 ft. base, with base angles each 45°. Calculate the maximum bending moment acting on the girder, assuming it to be freely supported at the ends. [259·2 tons-in.]

3. A propped cantilever beam 20 ft. long and of uniform cross-section carries a load of 2 tons per ft. The prop is 5 ft. from the free end and is at the same level as the fixed end. Find the load on the prop. Draw shearing-force and bending-moment diagrams for the beam.
[23·75 tons; S.F. at wall, 16·25 tons; max. B.M., 45 tons-ft. at wall.]

4. A beam 40 ft. long is supported at 10 ft. and 8 ft. from each end and is loaded with a uniformly distributed load of $\frac{1}{2}$ ton per foot run. Draw bending-moment and shear-force diagrams and find the points of contraflexure. What is the bending moment at the centre of the beam?
[15·5 ft. and 11·87 ft. from the ends; 9·1 tons-ft.]

5. A beam of uniform cross-section and of length l carries a uniformly distributed load of w per unit length; it is simply supported at the left-hand end and at a point $\frac{1}{3}l$ inside the right-hand end. Find in terms of E, I, w and l the deflection of the overhanging end. $\left[\frac{wl^4}{684EI}.\right]$

6. A horizontal steel beam, simply supported at the ends, carries a load which varies uniformly from 0·6 ton-ft. at one end to 1·6 tons-ft. at the other end over a span of 16 ft. Find the magnitude of the maximum bending moment.

If the depth of the beam is 16 in. and the maximum bending stress is 6 tons-in.^{2}, find the deflection at mid-span. (London Univ.)

[35·45 tons-ft.; 0·23 in.]

7. A beam AB, 20 ft. span, is supported and loaded in such a manner that the bending moment at any section distance x from one support is given by the equation $M = 10x - 0{\cdot}025x^3$ tons-ft.

Determine (*a*) the nature and magnitude of the load on the beam, and (*b*) the pressure on the supports.

[(*a*) 0 to 3 tons per ft.; total 30 tons. (*b*) 10 tons; 20 tons.]

✓8. A beam freely supported over a span of 20 ft. carries a load which varies uniformly from zero at one end to an intensity of 3 tons per ft. at the other. If the second moment of area of the beam section is 300 in.^{4} units, and the modulus of elasticity is 30×10^6 lb. per sq. in., calculate the deflection at the centre of the span. [1·344 in.]

9. A horizontal beam is simply supported at two points distance l apart and overhangs each support by $l/5$. Concentrated loads, each of magnitude W, are carried at each end of the beam. Find the magnitude of the uniformly distributed load carried between the two supports which would reduce the central deflection to zero. Find, for these conditions of loading, the end deflections in terms of W, l, and the usual constants. Neglect the weight of the beam. (London Univ.) [$4{\cdot}1W$; $17Wl^3/750EI$.]

10. A vertical column 20 ft. high is fixed at the base and a clockwise moment of 1000 lb.-ft. is applied at the top of the column. A horizontal force of P lb. is applied to the column at a height of 10 ft. above the base so as to give a counter-clockwise moment. Determine the value of P so that the horizontal deflections at the top of the column and at the point of application of P shall be equal, (*a*) when the deflections are on the same side, (*b*) when the deflections are on opposite sides of the vertical line through the foot of the column. (London Univ.)

[(*a*) 34·6 tons; (*b*) 83·3 tons.]

11. A girder 30 ft. long carrying a uniformly distributed load of w tons per ft. is to be supported on two piers 18 ft. apart so that the greatest bending moment on the girder shall be as small as possible. Find the distances of the piers from the ends of the girder and the maximum bending moment. (London Univ.)

12. A uniform beam is supported at its ends and carries a uniformly distributed load along the middle half. Show that the additional deflection due to the load is 57/64 times the additional deflection had the load been concentrated at the mid-point. (London Univ.)

13. Two vertical slings are to be used to raise a newly cast reinforced-concrete pile of length L and uniform cross-section, the pile remaining horizontal during the lift. Determine the most suitable positions for the

slings and sketch the shearing-force and bending-moment diagrams when the slings are in this position.

[$\frac{1}{2}(\sqrt{2} - 1)L$ from each end; max. B.M. $= \frac{1}{8}(1 - 2\sqrt{2})wL^2$; max. S.F. $= \pm\frac{1}{2}(\sqrt{2} - 1)wL$ and $\pm\frac{1}{2}w\{L - (\sqrt{2} - 1)L\}$].

14. A horizontal beam of uniform section and length L rests on supports at its ends. It carries a uniform load w per unit length, which extends over a length l from the right-hand support. Determine the value of l in order that the maximum deflection may occur at the left-hand end of the load, and if the maximum deflection is $wL^4/(kEI)$, find the value of k. (London Univ.) [$0{\cdot}451L$; 177.]

15. A horizontal beam of uniform section is simply supported over a span L and carries a uniformly distribted load w per unit length over the whole span. The distributed load is to be replaced by three concentrated loads each $\frac{1}{3}wL$, equally spaced, with the end loads each at a distance l from the nearer support. Find the values of l so that, for both types of loading, (a) the maximum bending moment is the same; (b) the maximum deflection is the same. (London Univ.) [$L/8$; $L/7{\cdot}1$.]

16. A beam of uniform cross-section is encastered horizontally at one end and is loaded with equal loads (W) at the free end and at distances a, $2a$, $3a$, etc., from the free end. Prove that the deflection at a distance na from the free end differs from the mean of the deflections at distances $(n - 1)a$ and $(n + 1)a$ by $\frac{Wa^3}{EI}\left(\frac{3n^2 + 3n + 1}{12}\right)$, where E and I have the usual significance.

17. Taking the safe stresses for concrete and steel to be 500 and 10,000 lb. per sq. in. respectively and their ratio of elasticity as 10, find the distance from the top of the flange of a tee beam the steel must be placed if the neutral axis of the section lies along the junction of the web and flange. If two steel rods are used for reinforcement, what must be their diameter if the span of the beam is 20 ft.? What is the value of the central deflection? E for concrete $= 3 \times 10^6$ lb. per sq. in. The flange is 15 in. by 5 in. [15 in.; $1\frac{1}{8}$ in.; 0·2 in.]

18. A reinforced-concrete beam is rectangular in section and has a width of 5 in. There are two steel bars of $\frac{7}{16}$ in. diameter placed at $8\frac{1}{2}$ in. from the top of the beam. If the maximum compressive stress in the concrete is to be 500 lb. per sq. in. and the modulii are 2×10^6 and 30×10^6 lb. per sq. in. for concrete and steel respectively, find the value of the maximum deflection when the beam supports a uniformly distributed load over a 10 ft. span. [0·12 in.]

19. A reinforced-concrete beam of rectangular cross-section is required to span 20 ft. and to support a load of 1200 lb. per ft. If the width of the beam is 40 per cent of the depth to the reinforcement, determine its size. The tensile stress allowed in the steel is 10,000 lb. per sq. in. and in the concrete 500 lb. per sq. in. E for steel $= 30 \times 10^6$ lb. per sq. in.; and E for concrete $= 2{\cdot}5 \times 10^6$ lb. per sq. in. What is the value of the central deflection? [$11\frac{1}{4}$ in. by 28 in.; 0·14 in.]

CHAPTER III

Encastré Beams

1. Beams with Various Forms of Loading.

When the beam is fixed at both ends so that the slopes of the ends of the beam are zero, there must be a fixing moment which gives this condition.



Fig. 3.1

Example 1.—A beam of span L fixed at both ends has a concentrated load W at the centre. The supporting forces at A and B must be each equal to $\frac{1}{2}W$ from symmetry, and at each end a fixing moment must operate in order to give zero slope. Thus the bending moment at XX $= \frac{1}{2}Wx - M$,

i.e.
$$EI\frac{d^2y}{dx^2} = \frac{1}{2}Wx - M,$$

$$\therefore\ EI\frac{dy}{dx} = \frac{1}{4}Wx^2 - Mx + A,$$

$$\therefore\ EIy = \frac{1}{12}Wx^3 - \frac{1}{2}Mx^2 + Ax + B.$$

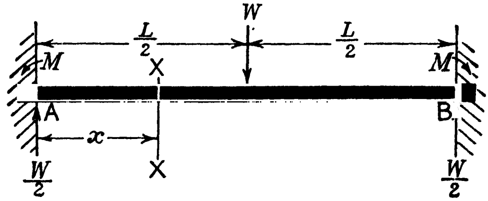


Fig. 3.2

Known conditions:

$$y = 0 \text{ when } x = 0, \quad \ldots \quad \text{(i)}$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0, \quad \ldots \quad \text{(ii)}$$

$$\frac{dy}{dx} = 0 \text{ when } x = \tfrac{1}{2}l. \quad \ldots \quad \text{(iii)}$$

Substituting condition (i), $B = 0.$

Substituting condition (ii), $A = 0,$

and from (iii), $$EI\frac{dy}{dx} = 0 = \tfrac{1}{16}Wl^2 - \tfrac{1}{2}Ml.$$

$$\therefore\ M = \tfrac{1}{8}Wl.$$

Hence bending moment at any section $= \tfrac{1}{2}Wx - \tfrac{1}{8}Wl,$

$$\text{slope} = \frac{1}{EI}\left\{\tfrac{1}{4}Wx^2 - \tfrac{1}{8}Wlx\right\},$$

$$\text{deflection} = \frac{1}{EI}\left\{\tfrac{1}{12}Wx^3 - \tfrac{1}{16}Wlx^2\right\}.$$

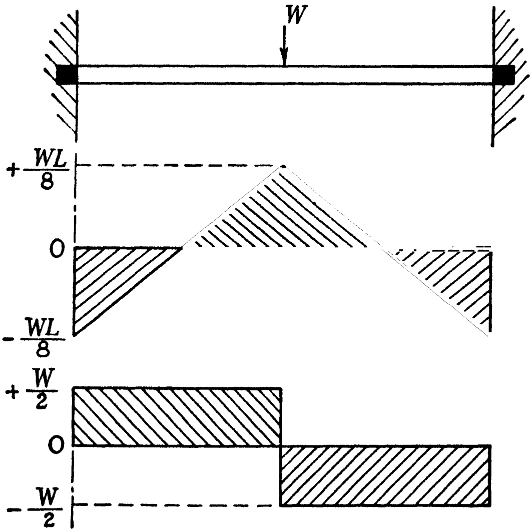


Fig. 3.3

Maximum deflection occurs when $x = \tfrac{1}{2}l$, and

$$= \frac{1}{EI}\left\{\tfrac{1}{12}W(\tfrac{1}{2}l)^3 - \tfrac{1}{16}Wl(\tfrac{1}{2}l)^2\right\}$$

$$= -\tfrac{1}{192}\frac{Wl^3}{EI}.$$

The shearing-force diagram will not be changed by the fixing, since it is the algebraic sum of the forces.

The bending-moment and shearing-force diagrams are shown in fig. 3.3.

Example 2.—A fixed beam has a span L with uniformly distributed load. Find expressions for the shearing force, bending moment, slope and deflection for any section of the beam.

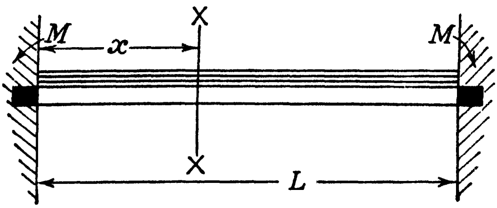


Fig. 3.4

For section XX, $EI\frac{d^4y}{dx^4} = -w.$

$$\therefore\ EI\frac{d^3y}{dx^3} = -wx + A.$$

$$\therefore\ EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}wx^2 + Ax + B.$$

$$\therefore\ EI\frac{dy}{dx} = -\tfrac{1}{6}wx^3 + \tfrac{1}{2}Ax^2 + Bx + C.$$

$$\therefore\ EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2 + Cx + D.$$

Known conditions: $y = 0$ when $x = 0$ and L,

$\frac{dy}{dx} = 0$ when $x = 0$, $\tfrac{1}{2}L$ and L.

Substituting $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$, then

$$C = 0 \text{ and } D = 0.$$

When $x = L$, $y = 0$,

$$\therefore\ EIy = 0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{6}AL^3 + \tfrac{1}{2}BL^2.$$

$$\therefore\ B = \tfrac{1}{12}wL^2 - \tfrac{1}{3}AL. \quad \ldots\ldots\ldots \quad (1)$$

Also $dy/dx = 0$ when $x = L$,

$$\therefore\ EI\frac{dy}{dx} = 0 = -\tfrac{1}{6}wL^3 + \tfrac{1}{2}AL^2 + BL.$$

$$\therefore\ B = \tfrac{1}{6}wL^2 - \tfrac{1}{2}AL. \quad \ldots\ldots\ldots \quad (2)$$

Equating (1) and (2),

$$\tfrac{1}{12}wL^2 - \tfrac{1}{3}AL = \tfrac{1}{6}wL^2 - \tfrac{1}{2}AL.$$

$$\therefore\ \tfrac{1}{6}AL = \tfrac{1}{12}wL^2,$$

$$\therefore\ A = \tfrac{1}{2}wL,$$

and $$B = \tfrac{1}{6}wL^2 - \tfrac{1}{4}wL^2 = -\tfrac{1}{12}wL^2.$$

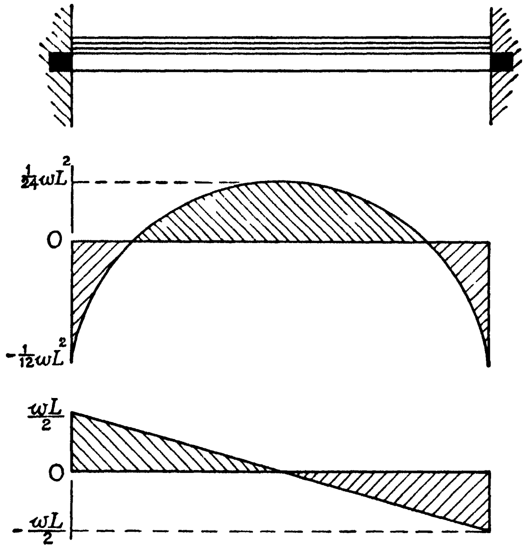


Fig. 3.5

Then $$EI\frac{d^4y}{dx^4} = -w.$$

$$\therefore\ EI\frac{d^3y}{dx^3} = -wx + \tfrac{1}{2}wL.$$

$$\therefore\ EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}wx^2 + \tfrac{1}{2}wLx - \tfrac{1}{12}wL^2.$$

$$\therefore\ EI\frac{dy}{dx} = -\tfrac{1}{6}wx^3 + \tfrac{1}{4}wLx^2 - \tfrac{1}{12}wL^2x.$$

$$\therefore\ EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^2x^2.$$

Maximum deflection occurs when $x = \tfrac{1}{2}L$ and is

$$\frac{1}{EI}\left\{-\frac{wL^4}{384} + \frac{wL^4}{96} - \frac{wL^4}{96}\right\} = -\tfrac{1}{384}\frac{wL^4}{EI}.$$

The shearing-force and bending-moment diagrams are as shown in fig. 3.5.

Example 3.—A beam of span L is fixed at both ends and carries a load which varies from zero at the left-hand end to w per unit run at the right-hand end. Find expressions for the shearing force, bending moment, slope and deflection of any section of the beam. Sketch the shearing-force and bending-moment diagrams, deduce an expression for the maximum deflection and find where it occurs.

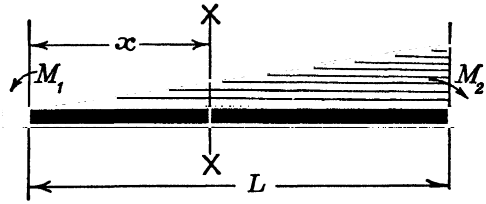


Fig. 3.6

$$EI\frac{d^4y}{dx^4} = -s = -\frac{wx}{L}.$$

$$EI\frac{d^3y}{dx^3} = -\frac{wx^2}{2L} + A.$$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6L} + Ax + B.$$

$$EI\frac{dy}{dx} = -\frac{wx^4}{24L} + \tfrac{1}{2}Ax^2 + Bx + C.$$

$$EIy = -\frac{wx^5}{120L} + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2 + Cx + D.$$

Known conditions: $y = 0$ when $x = 0$, $\therefore D = 0$.

$\frac{dy}{dx} = 0$ when $x = 0$, $\therefore C = 0$.

$y = 0$ when $x = L$,

$$\therefore 0 = -\frac{wL^5}{120L} + \tfrac{1}{6}AL^3 + \tfrac{1}{2}BL^2. \quad \ldots\ldots \text{(i)}$$

$\frac{dy}{dx} = 0$ when $x = L$,

$$\therefore 0 = -\frac{wL^4}{24L} + \tfrac{1}{2}AL^2 + BL. \quad \ldots\ldots \text{(ii)}$$

From (ii), $B = \tfrac{1}{24}wL^2 - \tfrac{1}{2}AL.$

Substituting in (i),

$$0 = -\frac{wL^5}{120L} + \tfrac{1}{6}AL^3 + \tfrac{1}{24}wL^2 \cdot \tfrac{1}{2}L^2 - \tfrac{1}{2}AL \cdot \tfrac{1}{2}L^2$$

$$= -\tfrac{1}{120}wL^4 + \tfrac{1}{48}wL^4 - \tfrac{1}{12}AL^3.$$

$$\therefore\ A = \tfrac{3}{20}wL,$$

and $B = \tfrac{1}{24}wL^2 - \tfrac{3}{40}wL^2 = -\tfrac{1}{30}wL^2 = -\tfrac{1}{15}WL^2,$

where $W = \tfrac{1}{2}wL$ = total load.

Hence

$$EI\frac{d^3y}{dx^3} = -\frac{wx^2}{2L} + \tfrac{3}{20}wL.$$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6L} + \tfrac{3}{20}wLx - \tfrac{1}{30}wL^2.$$

$$EI\frac{dy}{dx} = -\frac{wx^4}{24L} + \tfrac{3}{40}wLx^2 - \tfrac{1}{30}wL^2x.$$

$$EIy = -\frac{wx^5}{120L} + \tfrac{1}{40}wLx^3 - \tfrac{1}{60}wL^2x^2.$$

Shearing force when $x = 0 = \tfrac{3}{20}wL$;

when $x = L$ it is $-\dfrac{wL^2}{2L} + \tfrac{3}{20}wL = -\tfrac{7}{20}wL,$

and when $x = \tfrac{1}{2}L$ it is $-\dfrac{wL^2}{8L} + \tfrac{3}{20}wL = \tfrac{1}{40}wL.$

Bending moment when $x = 0$ is $-\tfrac{1}{30}wL^2$;

when $x = L$ it is $-\dfrac{wL^3}{6L} + \tfrac{3}{20}wL^2 - \tfrac{1}{30}wL^2 = -\tfrac{1}{20}wL^2,$

and when $x = \tfrac{1}{2}L$ it is $-\dfrac{wL^3}{48L} + \tfrac{3}{40}wL^2 - \tfrac{1}{30}wL^2 = \tfrac{1}{48}wL^2.$

Maximum deflection occurs when $\dfrac{dy}{dx} = 0$, i.e. when

$$-\frac{wx^4}{24L} + \tfrac{3}{40}wLx^2 - \tfrac{1}{30}wL^2x = 0,$$

or

$$\frac{x^4}{24L} - \tfrac{3}{40}Lx^2 + \tfrac{1}{30}L^2x = 0,$$

i.e.

$$x^3 - 1{\cdot}8L^2x + 0{\cdot}8L^3 = 0.$$

This equation is satisfied when $x = 0{\cdot}525L$.

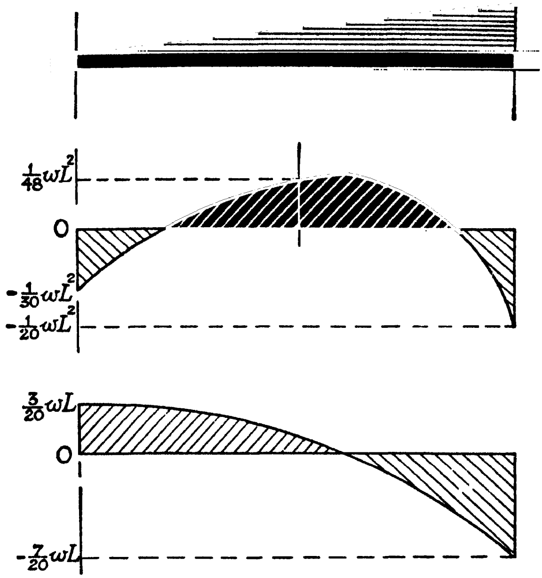


Fig. 3.7

Substituting this value,

$$EIy = -\frac{wx^5}{120L} + \tfrac{1}{40}wLx^3 - \tfrac{1}{60}wL^2x^2$$

$$= -\frac{wL^4}{120}(0{\cdot}525^5 - 3 \times 0{\cdot}525^3 + 2 \times 0{\cdot}525^2)$$

$$= -\frac{wL^4}{120}({\cdot}0398 - {\cdot}4341 + {\cdot}5513)$$

$$= -\frac{wL^4}{120} \times {\cdot}1570$$

$$= -\tfrac{1}{764}wL^4$$

$$= -\tfrac{1}{382}WL^3.$$

Example 4.—A beam of span L is fixed at both ends and carries a concentrated load at a distance a from the left-hand wall and b from the right-hand wall. Find expressions for bending moment, slope and deflection, the value of the fixing moment, the deflection under the load, position of maximum deflection and its value.

In order that the slope at the walls may be zero, there must be fixing moments M_1 and M_2 acting.

The external load W must be supported by the two walls. The reactions R_1 and R_2 may be found by taking moments and are

$$R_1 = \frac{Wb}{L} \text{ and } R_2 = \frac{Wa}{L}.$$

The bending moment at section XX due to the external load is

$$R_1x = \frac{Wbx}{L}.$$

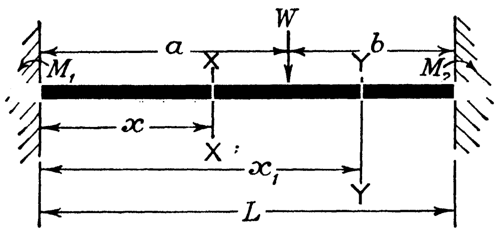


Fig. 3.8

The fixing-moment diagram will be a straight line joining the values M_1 and M_2 at the ends of the beam, hence the moment at XX due to the fixing moments will be

$$M_1 + \frac{(M_2 - M_1)}{L} x.$$

Hence total moment at XX is

$$\frac{Wbx}{L} - M_1 - (M_2 - M_1)\frac{x}{L}.$$

$$\therefore EI\frac{d^2y}{dx^2} = \frac{Wbx}{L} - M_1 - (M_2 - M_1)\frac{x}{L}$$

and $$EI\frac{dy}{dx} = \frac{Wbx^2}{2L} - M_1x - (M_2 - M_1)\frac{x^2}{2L} + A. \quad . \quad . \quad (1)$$

Since $\frac{dy}{dx} = 0$ when $x = 0$, $\therefore A = 0$.

$$\therefore EIy = \frac{Wbx^3}{6L} - \tfrac{1}{2}M_1x^2 - (M_2 - M_1)\frac{x^3}{6L} + B. \quad . \quad . \quad (2)$$

Since $y = 0$ when $x = 0$, $\therefore B = 0$.

In order to find the values of M_1 and M_2 it is necessary to make use of other known conditions. It is known that the slope and deflection under the load may be obtained by integrating the bending-moment expression over section a or over section b of the beam.

Let YY be a section between the load and 2. Then the bending moment at YY

$$= R_1 x_1 - W(x_1 - a) - M_1 - (M_2 - M_1)\frac{x_1}{L}$$

$$= \frac{Wbx_1}{L} - Wx_1 + Wa - M_1 - (M_2 - M_1)\frac{x_1}{L}$$

$$= \left(\frac{Wb}{L} - W\right)x_1 + (Wa - M_1) - (M_2 - M_1)\frac{x_1}{L}$$

$$\therefore\ EI\frac{d^2y}{dx^2} = \left(\frac{Wb}{L} - W\right)x_1 + (Wa - M_1) - (M_2 - M_1)\frac{x_1}{L}.$$

$$EI\frac{dy}{dx} = \left(\frac{Wb}{L} - W\right)\tfrac{1}{2}x_1^2 + (Wa - M_1)x_1 - (M_2 - M_1)\frac{x_1^2}{2L} + A_1.$$

$$EIy = \left(\frac{Wb}{L} - W\right)\tfrac{1}{6}x_1^3 + (Wa - M_1)\tfrac{1}{2}x_1^2 - (M_2 - M_1)\frac{x_1^3}{6L} + A_1 x_1 + B_1.$$

Known conditions are: $y = 0$ when $x_1 = L$,

$$\frac{dy}{dx} = 0 \text{ when } x_1 = L.$$

$$\therefore\ EI\frac{dy}{dx} = 0 = \left(\frac{Wb}{L} - W\right)\tfrac{1}{2}L^2 + (Wa - M_1)L - (M_2 - M_1)\frac{L^2}{2L} + A_1.$$

$$\therefore\ A_1 = -\left[\left(\frac{Wb}{L} - W\right)\tfrac{1}{2}L^2 + W(L - b)L - M_1 L - \tfrac{1}{2}M_2 L + \tfrac{1}{2}M_1 L\right]$$

$$= (M_1 + M_2)\tfrac{1}{2}L - \tfrac{1}{2}WaL.$$

$$EIy = 0 = \left(\frac{Wb}{L} - W\right)\tfrac{1}{6}L^3 + (Wa - M_1)\tfrac{1}{2}L^2 - (M_2 - M_1)\frac{L^3}{6L} + (M_1 + M_2)\tfrac{1}{2}L^2 - \tfrac{1}{2}WaL^2 + B_1.$$

$$\therefore\ -B_1 = \tfrac{1}{6}WbL^2 - \tfrac{1}{6}WL^3 + \tfrac{1}{2}WaL^2 - \tfrac{1}{2}M_1L^2 - \tfrac{1}{6}M_2L^2 + \tfrac{1}{6}M_1L^2 + \tfrac{1}{2}M_1L^2 + \tfrac{1}{2}M_2L^2 - \tfrac{1}{2}WaL^2.$$

$$\therefore\ B_1 = \tfrac{1}{6}WaL^2 - \tfrac{1}{6}M_1L^2 - \tfrac{1}{3}M_2L^2.$$

Hence

$$EI\frac{dy}{dx} = \left(\frac{Wb}{L} - W\right)\tfrac{1}{2}x_1^2 + (Wa - M_1)x_1 - (M_2 - M_1)\frac{x_1^2}{2L} + (M_1 + M_2)\tfrac{1}{2}L - \tfrac{1}{2}WaL, \quad (3)$$

and
$$EIy = \left(\frac{Wb}{L} - W\right)\tfrac{1}{6}x_1^3 + (Wa - M_1)\tfrac{1}{2}x_1^2 - (M_2 - M_1)\frac{x_1^3}{6L} + (M_1 + M_2)\tfrac{1}{2}Lx_1 - \tfrac{1}{2}WaLx_1 + \tfrac{1}{6}WaL^2 - \tfrac{1}{6}M_1L^2 - \tfrac{1}{3}M_2L^2. \quad (4)$$

The slope under the load as given by equations (1) and (3) is equal, *i.e.* $x = x_1 = a$.

$$\therefore EI\frac{dy}{dx} = \left(\frac{Wb}{L} - W\right)\tfrac{1}{2}a^2 + (Wa - M_1)a - (M_2 - M_1)\frac{a^2}{2L} + (M_1 + M_2)\tfrac{1}{2}L - \tfrac{1}{2}WaL$$

$$= \frac{Wba^2}{2L} - M_1a - (M_2 - M_1)\frac{a^2}{2L}.$$

$$\therefore 0 = \tfrac{1}{2}Wa^2 - \tfrac{1}{2}WaL + \tfrac{1}{2}M_1L + \tfrac{1}{2}M_2L.$$

$$\therefore (M_1 + M_2) = \frac{Wab}{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The deflection under the load as given by equations (2) and (4) is equal, i.e. $x = x_1 = a$.

$$\therefore EIy = \left(\frac{Wb}{L} - W\right)\tfrac{1}{6}a^3 + (Wa - M_1)\tfrac{1}{2}a^2 - (M_2 - M_1)\frac{a^3}{6L} + (M_1 + M_2)\tfrac{1}{2}La - \tfrac{1}{2}Wa^2L + \tfrac{1}{6}WaL^2 - \tfrac{1}{6}M_1L^2 - \tfrac{1}{3}M_2L^2$$

$$= \frac{Wba^3}{6L} - \tfrac{1}{2}M_1a^2 - (M_2 - M_1)\frac{a^3}{6L}.$$

$$\therefore 0 = \tfrac{1}{3}Wa^3 - \tfrac{1}{2}Wa^2L + \tfrac{1}{6}WaL^2 + M_1(\tfrac{1}{2}aL - \tfrac{1}{6}L^2) + M_2(\tfrac{1}{2}aL - \tfrac{1}{3}L^2).$$

But $M_1 = \dfrac{Wab}{L} - M_2$ from (5).

$$\therefore 0 = \tfrac{1}{3}Wa^3 - \tfrac{1}{2}Wa^2L + \tfrac{1}{6}WaL^2 + \tfrac{1}{2}Wa^2b - \tfrac{1}{6}WabL - \tfrac{1}{2}M_2aL + \tfrac{1}{6}M_2L^2 + \tfrac{1}{2}M_2aL - \tfrac{1}{3}M_2L^2$$

$$= \tfrac{1}{3}Wa^3 - \tfrac{1}{2}Wa^2(L - b) + \tfrac{1}{6}WaL(L - b) - \tfrac{1}{6}M_2L^2$$

$$= -\tfrac{1}{6}Wa^3 + \tfrac{1}{6}Wa^2L - \tfrac{1}{6}M_2L^2$$

$$= \tfrac{1}{6}Wa^2b - \tfrac{1}{6}M_2L^2.$$

$$\therefore M_2 = \frac{Wa^2b}{(a + b)^2} = \text{fixing moment at end 2,}$$

and

$$M_1 = \frac{Wab}{L} - \frac{Wa^2b}{(a + b)^2}$$

$$= \frac{Wab^2}{(a + b)^2} = \text{fixing moment at end 1.}$$

It will be noticed that the average fixing moment

$$= \frac{Wab^2 + Wa^2b}{2(a + b)^2} = \frac{Wab}{2(a + b)}.$$

Thus the area of the bending-moment diagram due to M_1 and M_2 is

$$\tfrac{1}{2}\frac{Wab}{a+b}(a+b) = \tfrac{1}{2}Wab.$$

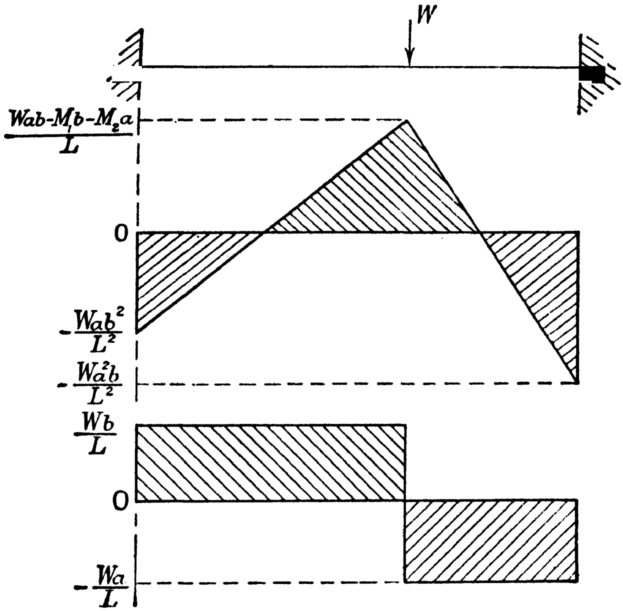


Fig. 3.9

Now the area of the bending-moment diagram due to the external load

$$= \text{area } xyz \text{ in fig. 3.10}$$

$$= \tfrac{1}{2}\cdot\frac{Wba}{L}\cdot a + \tfrac{1}{2}\frac{Wba}{L}\cdot b$$

$$= \tfrac{1}{2}\frac{Wab}{L}(a+b)$$

$$= \tfrac{1}{2}Wab$$

$$= \text{area of fixing-moment diagram.}$$

The area of the bending-moment diagram due to the external load will always equal the area of the fixing-moment diagram in the case of fixed beams, since the slope at both ends is zero and slope is the integral of the bending-moment expression.

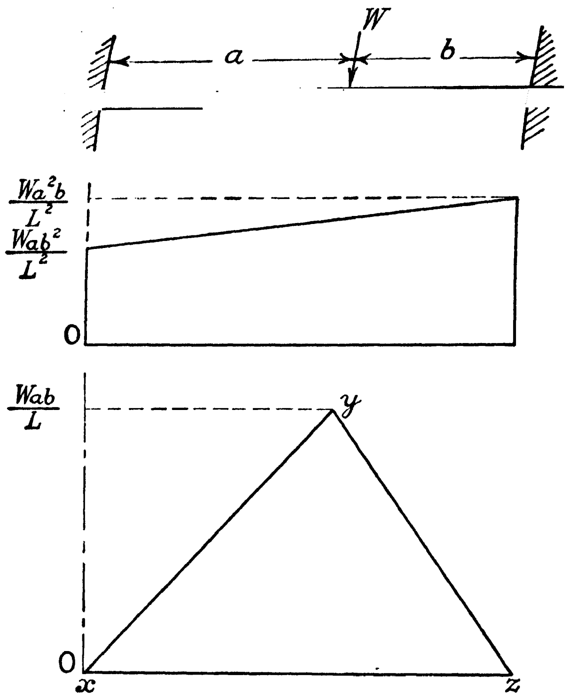


Fig. 3.10

Deflection under load is given by equation (2).

$$EIy = \frac{Wba^3}{6L} - \frac{Wab^2}{L^2} \cdot \frac{a^2}{2} - \left(\frac{Wa^2b}{L^2} - \frac{Wab^2}{L^2}\right)\frac{a^3}{6L}$$

$$= \frac{Wba^3}{6L} - \frac{Wa^3b^2}{2L^2} - \frac{Wa^5b}{6L^3} + \frac{Wa^4b^2}{6L^3}$$

$$= \frac{Wa^3b}{6L^3}(L^2 - 3bL - a^2 + ab)$$

$$= \frac{Wa^3b}{6L^3}[(a + b)^2 - 3b(a + b) - a^2 + ab]$$

$$= \frac{Wa^3b}{6L^3}(-2b^2)$$

$$= -\frac{Wa^3b^3}{3L^3}$$

$$= -\frac{Wa^3b^3}{3(a + b)^3}.$$

The maximum deflection will occur where the slope is zero. If a is greater than b, then the maximum deflection will occur on the a portion of

the beam. The slope expression for that part of the beam is given by equation (1) and is

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{Wbx^2}{2L} - M_1x - \frac{(M_2 - M_1)x^2}{2L}\right]$$

$$= \frac{1}{EI}\left[\frac{Wbx^2}{2L} - \frac{Wab^2x}{L^2} - \frac{Wab(a-b)x^2}{2L^3}\right].$$

$$\therefore \frac{Wbx^2}{2L} - \frac{Wab^2x}{L^2} - \frac{Wab(a-b)x^2}{2L^3} = 0,$$

or

$$\frac{Wbx}{2L^3}(L^2x - 2abL - a^2x + abx) = 0.$$

$$\therefore L^2x - 2abL - a^2x + abx = 0,$$

or

$$x(a^2 + 2ab + b^2) - 2a^2b - 2ab^2 - a^2x + abx = 0;$$

i.e.

$$3abx + b^2x - 2a^2b - 2ab^2 = 0.$$

$$\therefore (3ax + bx) = 2a(b + a),$$

or

$$x = \frac{2a(a+b)}{3a+b}.$$

For maximum deflection, substitute for x in equation (2).

$$EIy = \frac{Wbx^3}{6L} - \frac{Wab^2}{L^2}\cdot\frac{x^2}{2} - \frac{Wab(a-b)x^3}{6L^3}$$

$$= \frac{Wbx^2}{6L^3}[L^2x - 3abL - a(a-b)x]$$

$$= \frac{Wb}{6L^3}\frac{4a^2 . L^2}{(3a+b)^2}\left[\frac{L^2 . 2aL}{3a+b} - 3abL - a(a-b)\frac{2aL}{3a+b}\right]$$

$$= \frac{2}{3}\frac{Wa^2b}{(3a+b)^3}[2aL^2 - 9a^2b - 3ab^2 - 2a^3 + 2a^2b]$$

$$= \frac{2}{3}\frac{Wa^2b}{(3a+b)^3}[2a(a^2 + 2ab + b^2) - 7a^2b - 3ab^2 - 2a^3]$$

$$= \frac{2}{3}\frac{Wa^2b}{(3a+b)^3}[-3a^2b - ab^2]$$

$$= -\frac{2}{3}\frac{Wa^3b^2}{(3a+b)^2}.$$

$$\therefore y = -\frac{2}{3}\frac{Wa^3b^2}{(3a+b)^2EI}.$$

2. Effect of Sinking of Support.

If one of the supports sinks there will be an alteration in the bending-moment and shearing-force diagrams. The two ends will still have zero slope, but there will be a vertical displacement equal to the

amount of sinking between the two ends. It is therefore possible to solve the problem in the normal manner and find the values of the constants of integration for the given conditions.

Example 5.—A horizontal beam is fixed at both ends and supports a uniformly distributed load w per unit length. The span is L. If the right-hand end sinks by an amount e and the ends of the beam remain parallel, find expressions for the shearing force, bending moment, slope and deflection.

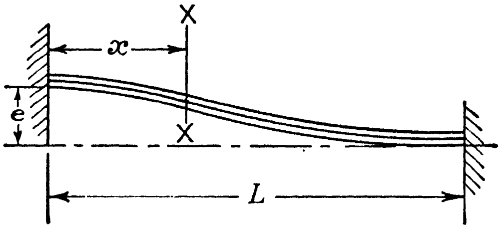


Fig. 3.11

For section XX,

$$EI\frac{d^4y}{dx^4} = -w.$$

$$EI\frac{d^3y}{dx^3} = -wx + A.$$

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}wx^2 + Ax + B.$$

$$EI\frac{dy}{dx} = -\tfrac{1}{6}wx^3 + \tfrac{1}{2}Ax^2 + Bx + C.$$

$$EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2 + Cx + D.$$

Known conditions:

$$y = 0 \text{ when } x = 0, \text{ and } y = -e \text{ when } x = L;$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0 \text{ and } L.$$

Substituting $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$, we have

$$C \text{ and } D = 0.$$

When $x = L$, $y = -e$. The negative sign must be used since the deflection is downwards.

Then $$EIy = -EIe = -\tfrac{1}{24}wL^4 + \tfrac{1}{6}AL^3 + \tfrac{1}{2}BL^2.$$

$$\therefore B = -\frac{2EIe}{L^2} + \tfrac{1}{12}wL^2 - \tfrac{1}{3}AL. \quad . \quad . \quad . \quad . \quad (1)$$

Also $\frac{dy}{dx} = 0$ when $x = L$,

$$\therefore\ EI\frac{dy}{dx} = 0 = -\tfrac{1}{6}wL^3 + \tfrac{1}{2}AL^2 + BL.$$

$$\therefore\ B = \tfrac{1}{6}wL^2 - \tfrac{1}{2}AL. \qquad (2)$$

Equating (1) and (2),

$$-\frac{2EIe}{L^2} + \tfrac{1}{12}wL^2 - \tfrac{1}{3}AL = \tfrac{1}{6}wL^2 - \tfrac{1}{2}AL.$$

$$\therefore\ \tfrac{1}{6}AL = \tfrac{1}{12}wL^2 + \frac{2EIe}{L^2},$$

$$\therefore\ A = \tfrac{1}{2}wL + \frac{12EIe}{L^3},$$

and

$$B = \tfrac{1}{6}wL^2 - \tfrac{1}{4}wL^2 - \frac{6EIe}{L^2}$$

$$= -\tfrac{1}{12}wL^2 - \frac{6EIe}{L^2}.$$

Then $EI\frac{d^4y}{dx^4} = -w.$

$$EI\frac{d^3y}{dx^3} = -wx + \tfrac{1}{2}wL + \frac{12EIe}{L^3}.$$

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}wx^2 + \tfrac{1}{2}wLx + \frac{12EIex}{L^3} - \tfrac{1}{12}wL^2 - \frac{6EIe}{L^2}.$$

$$EI\frac{dy}{dx} = -\tfrac{1}{6}wx^3 + \tfrac{1}{4}wLx^2 + \frac{6EIex^2}{L^3} - \tfrac{1}{12}wL^2x - \frac{6EIex}{L^2}.$$

$$EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 + \frac{2EIex^3}{L^3} - \tfrac{1}{24}wL^2x^2 - \frac{3EIex^2}{L^2}.$$

The bending moment when $x = 0$ is

$$-\tfrac{1}{12}wL^2 - \frac{6EIe}{L^2},$$

and when x is L, the bending moment

$$= -\tfrac{1}{2}wL^2 + \tfrac{1}{2}wL^2 + \frac{12EIeL}{L^3} - \tfrac{1}{12}wL^2 - \frac{6EIe}{L^2}$$

$$= \frac{6EIe}{L^2} - \tfrac{1}{12}wL^2.$$

The shearing force when $x = 0$ is

$$\tfrac{1}{2}wL + \frac{12EIe}{L^3},$$

and, when $x = L$, it is

$$-wL + \tfrac{1}{2}wL + \frac{12EIe}{L^3} = -\tfrac{1}{2}wL + \frac{12EIe}{L^3}.$$

Hence the bending or fixing moment is increased at the high end and diminished at the low end. Similarly there is an increase and

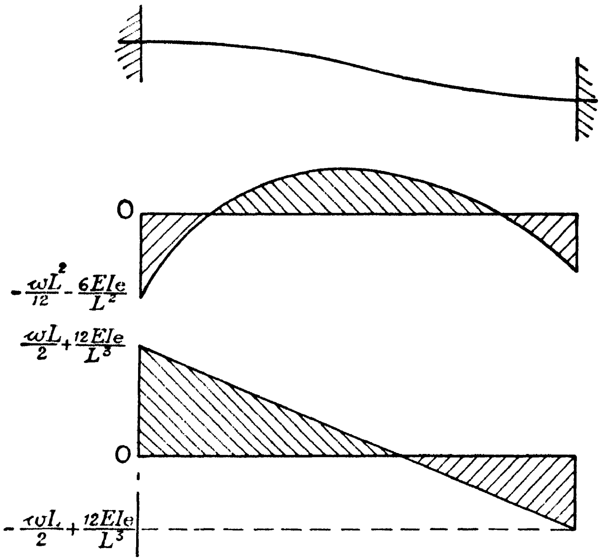


Fig. 3.12

decrease in the shearing forces at the high and low ends respectively.

It is possible to arrive at the same conclusions if the effect of the sinking of one end is obtained irrespective of the loading. In order

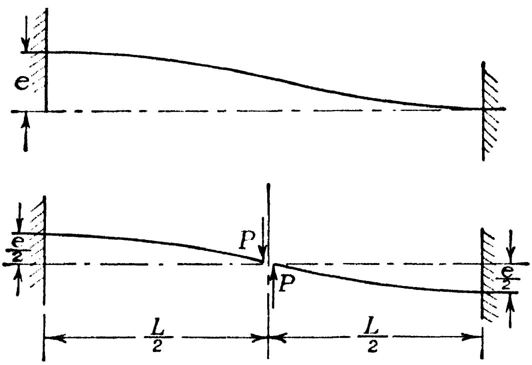


Fig. 3.13

that the one end of the beam might sink it is necessary for the beam to take up the shape shown in fig. 3.13. It is seen that the curve of flexure changes, therefore there is zero bending moment at the point

of change. From symmetry it can be assumed that the point of zero bending moment is midway between the two fixings. Assume that the beam is cut at that point, two cantilevers are then obtained (see fig. 3.13). The curve of flexure of the one cantilever is downwards whilst the other is upwards, and from the symmetrical nature of the problem the deflection of the end of each cantilever will be equal to one half the difference in the fixing levels. If a downward load P is applied to the end of the one cantilever and an upward load P to the other, then the static conditions remain the same, since equal and opposite forces have been applied at the same point. The downward and upward deflection of the cantilevers may then be considered to be the result of the loads P. The deflection of the end of a cantilever of length $\frac{1}{2}L$ when carrying a concentrated load P at the free end is

$$\frac{1}{3}\frac{P(\frac{1}{2}L)^3}{EI}.$$

But the deflection of the end of each cantilever due to the sinking is $\frac{1}{2}e$.

Hence $$\frac{1}{3}\frac{P(\frac{1}{2}L)^3}{EI}=\frac{e}{2},$$

$$\therefore\ P=\frac{12EIe}{L^3}.$$

The bending moment at the fixed ends due to P is

$$P\times\tfrac{1}{2}L=\frac{12EIe}{L^3}\times\frac{L}{2}$$

$$=\frac{6EIe}{L^2}.$$

Hence there is a variation of $12EIe/L^3$ in the shearing forces at the fixed ends and a variation in the fixing moments of $6EIe/L^2$. Shearing of the type shown in fig. 3.14 is considered to be positive and, from consideration of the conditions on the cantilever due to P, it is seen to be positive shearing in both cases. As the conventions used give a positive shearing force at the left end and a negative shearing force at the right end when the beam is subjected to downward loading, it is seen that the shearing force is increased at the high end and decreased at the low end due to the sinking (see fig. 3.15, p. 70).

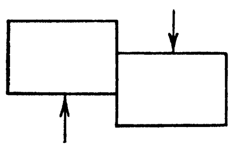


Fig. 3.14

The bending of the cantilever on the left is considered to be negative, whilst that on the right is positive. By the normal conventions used it is seen that the bending or fixing moment of a fixed beam

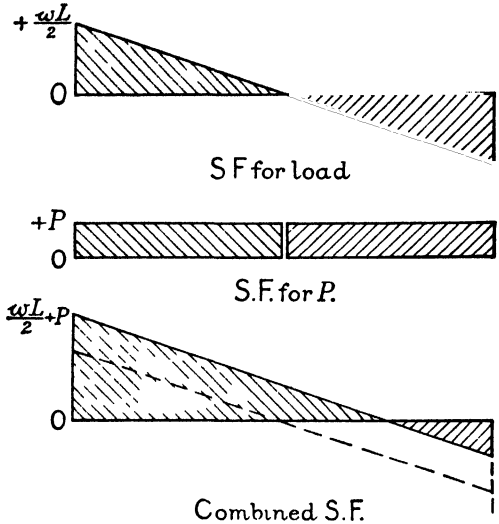


Fig. 3.15

with a downward load is negative at both ends, so that the fixing moment is increased on the high side and decreased on the low (fig. 3.16).

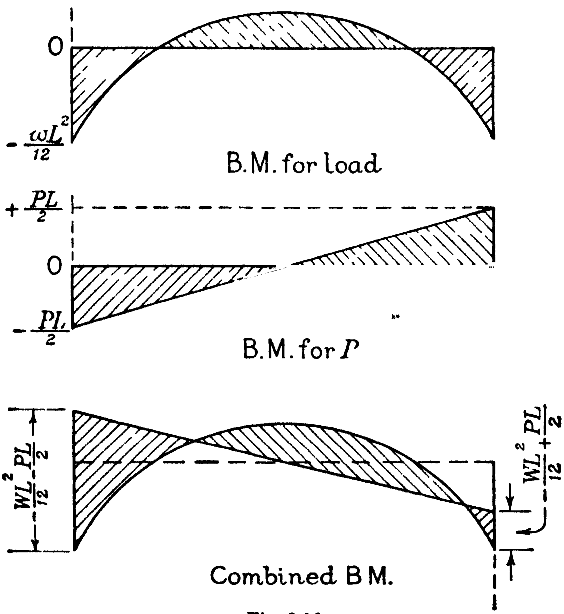


Fig. 3.16

Example 6.—A horizontal timber beam, 10 in. wide and 12. in deep, is fixed at each end. It supports a load of 12 tons at the centre. If the right-hand support sinks $\frac{1}{5}$ in. and the ends remain horizontal, calculate the maximum bending stress. The span is 24 ft. and the modulus of elasticity 1300 tons per sq. in.

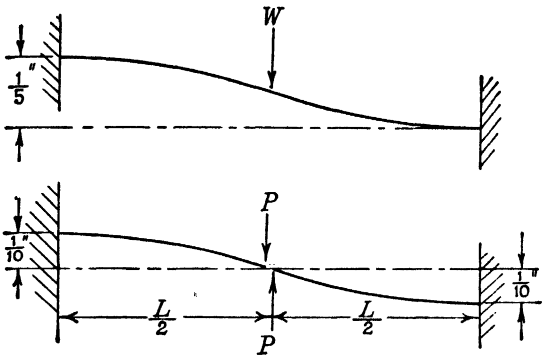


Fig. 3.17

Before the right-hand support sinks, the fixing moments are each equal to $\frac{1}{8}WL$, where $W = 12$ tons and $L = 24$ ft. The increase of the fixing moment due to the sinking is $6EIe/L^2$, where $e = \frac{1}{5}$ in., and E and I are the modulus of elasticity and the second moment of the area respectively. Hence maximum fixing moment

$$= \tfrac{1}{8}WL + \frac{6EIe}{L^2}$$

$$= \tfrac{1}{8} \times 12 \times 24 \times 12 + \frac{6 \times 1300 \times \frac{1}{12} \times 10 \times 12^3 \times \frac{1}{5}}{(24 \times 12)^2}$$

$$= 432 + 27{\cdot}1$$

$$= 459{\cdot}1 \text{ tons-in.}$$

From $$\frac{M}{I} = \frac{f}{m},$$

$$f = \frac{459{\cdot}1 \times 6}{\frac{1}{12} \times 10 \times 12^3}$$

$$= 1{\cdot}913 \text{ tons per sq. in}$$

3. Disadvantage of Encastré Beams.

Whilst the bending moment on a fixed beam is considerably less than when the same beam is freely supported, it is seldom that a fixed beam is used in practice. The main disadvantage is due to the uncertainty of the end fixings remaining exactly the same when in use. If the one support sinks, then there is a change in the bending or fixing moment which will alter the bending stresses in the material.

EXAMPLES

1. A beam, rigidly built in at each end, has a clear span of 12 ft. It carries a single concentrated load of 12 tons at a distance of 3 ft. from one end. Determine the bending moments in the beam at the two ends and under the load. [$-6{\cdot}75$ tons-ft.; $-20{\cdot}25$ tons-ft.; $+10{\cdot}125$ tons-ft.]

2. A beam, securely fixed at the ends and of length l between the fixings, is subjected to a couple M at the centre which tends to bend the beam. Find the values of the fixing moments and end reactions.

Draw, approximately to scale, the bending-moment diagram for the beam, inserting the principal values of the bending moment. (London Univ.) [$M/4$; $3{\cdot}5M/L$; $M/2$.]

3. A beam of uniform section is built in at each end so as to have a clear span of 20 ft. It carries a uniform distributed load of $\frac{3}{4}$ ton-ft. on the left-hand half of the beam, together with a 12 ton load at 15 ft. from the left-hand end. Find the reactions and the fixing moments at the ends and draw a bending-moment diagram for the beam, inserting the principal values. (London Univ.)

[7·965 tons; 11·535 tons; 28·5 tons-ft.; 41·6 tons-ft.]

4. A horizontal beam carries a uniformly distributed load of $\frac{1}{2}$ ton per foot over its whole length. The beam has its ends securely built in, the clear span is 20 ft., and the second moment of area is 200 in.^{4} units. Later it is observed that one end of the beam has been displaced 0·4 in. vertically. Assuming the ends are still securely built in and horizontal, determine the alteration of the fixing moments and vertical reactions. $E = 30 \times 10^6$ lb. per sq. in. [$\pm$250,000 lb.-in.; $\pm$2083 lb.]

5. A horizontal girder AB of uniform section has its ends encastered, its free length being l. If the load W covers the portion of the beam distant a and b from A and B respectively, a being greater than b, show that for this position of the load the greatest deflection produced occurs at a section distant $\dfrac{2al}{2a+l}$ from A and is of magnitude $\dfrac{2a^3b^2W}{3(2a+l)^2EI}$.

6. An initially straight beam of uniform section ($I = 1000$ in.^{4}) is maintained horizontal at each end by being built into walls 20 ft. apart. The beam carries a uniformly distributed load of $1\frac{1}{2}$ tons per foot run over

the whole span, and a concentrated load of 10 tons at the centre of the span. One wall sinks 0·2 in. from its original position, but the end of the beam is still maintained horizontal. Sketch the bending-moment and shearing-force diagrams for this case, approximately to scale, and calculate the maximum values of the bending moment and shearing force. (London Univ.)

[22·22 tons; 17·78 tons; 1170 tons-in.; 630 tons-in.]

7. Given the bending-moment and shearing-force diagrams for the load system on a simply supported beam, explain how these diagrams would be modified to adapt them to the case where the beam is firmly fixed horizontally at its ends. Give illustrations and proof.

Obtain expressions for the moments of the end couples in the case of a beam fixed perfectly horizontally at its ends and carrying a system of concentrated loads.

CHAPTER IV

Continuous Beams

1. General Solution.

When a beam is supported by three or more supports it is known as a continuous beam. It has been shown in Chap. II, § 3 that a particular solution for a beam with three supports may be obtained. A general solution of the problem both for distributed and concentrated loads may be made in the following manner:

(a) Uniformly distributed load

Let all the supports be at the same level and take B as the origin and the axis of x along AB. Then

$$EI\frac{d^4y}{dx^4} = -w.$$

$$EI\frac{d^3y}{dx^3} = -wx + A_1.$$

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}wx^2 + A_1x + B_1.$$

$$EI\frac{dy}{dx} = -\tfrac{1}{6}wx^3 + \tfrac{1}{2}A_1x^2 + B_1x + C_1.$$

$$EIy = -\tfrac{1}{24}wx^4 + \tfrac{1}{6}A_1x^3 + \tfrac{1}{2}B_1x^2 + C_1x + D_1.$$

Known conditions:

bending moment $= M_b$ when $x = 0$, and M_a when $x = l_1$;

deflection $= 0$ when $x = 0$ and $x = l_1$.

$$\therefore\ EI\frac{d^2y}{dx^2} = M_b = 0 + 0 + B_1,\quad \therefore\ B_1 = M_b.$$

$$EI\frac{d^2y}{dx^2} = M_a = -\tfrac{1}{2}wl_1^{\,2} + A_1l_1 + M_b.$$

$$\therefore\ A_1 = \tfrac{1}{2}wl_1 - \frac{M_b}{l_1} + \frac{M_a}{l_1}.$$

$$EIy = 0 = 0 + D_1, \quad \therefore D_1 = 0.$$

$$\begin{aligned} EIy = 0 &= -\tfrac{1}{24}wl_1^4 + \tfrac{1}{6}A_1l_1^3 + \tfrac{1}{2}B_1l_1^2 + C_1l_1 \\ &= -\tfrac{1}{24}wl_1^4 + \tfrac{1}{12}wl_1^4 - \tfrac{1}{6}M_bl_1^2 + \tfrac{1}{6}M_al_1^2 + \tfrac{1}{2}M_bl_1^2 + C_1l_1 \\ &= \tfrac{1}{24}wl_1^4 + \tfrac{1}{3}M_bl_1^2 + \tfrac{1}{6}M_al_1^2 + C_1l_1. \end{aligned}$$

$$\therefore C_1 = -\tfrac{1}{24}wl_1^3 - \tfrac{1}{3}M_bl_1 - \tfrac{1}{6}M_al_1.$$

If the axis of x is along CB, then

$$C_2 = -\tfrac{1}{24}wl_2^3 - \tfrac{1}{3}M_bl_2 - \tfrac{1}{6}M_cl_2.$$

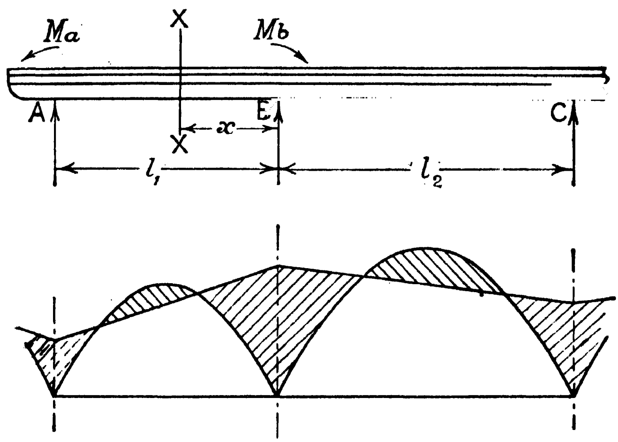


Fig. 4.1

But C_1 and C_2 will be the slope of the curve of flexure at B, i.e. when $x = 0$, and, since the directions of the axis of x are opposite in the two cases,

$$C_1 = -C_2.$$

$$\therefore -\tfrac{1}{24}wl_1^3 - \tfrac{1}{3}M_bl_1 - \tfrac{1}{6}M_al_1 = \tfrac{1}{24}wl_2^3 + \tfrac{1}{3}M_bl_2 + \tfrac{1}{6}M_cl_2.$$

$$\therefore \tfrac{1}{6}M_al_1 + \tfrac{1}{3}M_b(l_1 + l_2) + \tfrac{1}{6}M_cl_2 = -\tfrac{1}{24}wl_1^3 - \tfrac{1}{24}wl_2^3. \qquad (1)$$

This equation is generally known as Clapeyron's equation or the Theorem of Three Moments. If the beam only covers spans AB and BC, and is simply supported at the two ends, then M_a and M_b are zero.

In order to solve problems where the beam covers several spans, it is necessary to start at one end and work out equations to two spans at a time until all the spans have been considered. A number of simultaneous equations then have to be solved in order to effect a solution.

Example 1.- A beam ABC is simply supported at A, B and C, and is of length $2l$, each span being l. The beam carries a uniformly distributed load of w over its entire length; find the bending moment at B and sketch the bending-moment diagram. What are the values of the supporting forces at A, B and C?

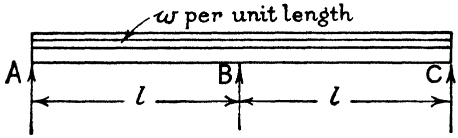


Fig 4.2

Since the beam is simply supported at A and C the moments at A and C will be zero.

Substituting in equation (1),

$$\tfrac{1}{6}M_a l_1 + \tfrac{1}{3}M_b(l_1 + l_2) + \tfrac{1}{6}M_c l_2 = -\tfrac{1}{24}wl_1^3 - \tfrac{1}{24}wl_2^3,$$

i.e.

$$0 + \tfrac{1}{3}M_b(l + l) + 0 = -\tfrac{1}{24}wl^3 - \tfrac{1}{24}wl^3.$$

$$\therefore M_b = -\tfrac{1}{8}wl^2.$$

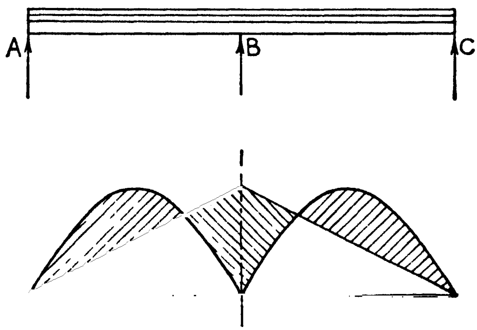


Fig. 4.3

Let R_a = reaction at A.

Then bending moment at B is

$$R_a l - wl \cdot \tfrac{1}{2}l = -\tfrac{1}{8}wl^2,$$

$$\therefore R_a = \tfrac{3}{8}wl.$$

Similarly $R_c = \tfrac{3}{8}wl$. But total load $= 2wl$.

$$\therefore R_b = 2wl - \tfrac{3}{4}wl$$

$$= 1\tfrac{1}{4}wl.$$

(*b*) *Continuous beam with single concentrated load on each span.*

Let M_a be the bending or fixing moment at A,

M_b be the bending or fixing moment at B,

M_c be the bending or fixing moment at C,

R_1 be the reaction at A,

R_2 be the reaction at B,

R_3 be the reaction at C.

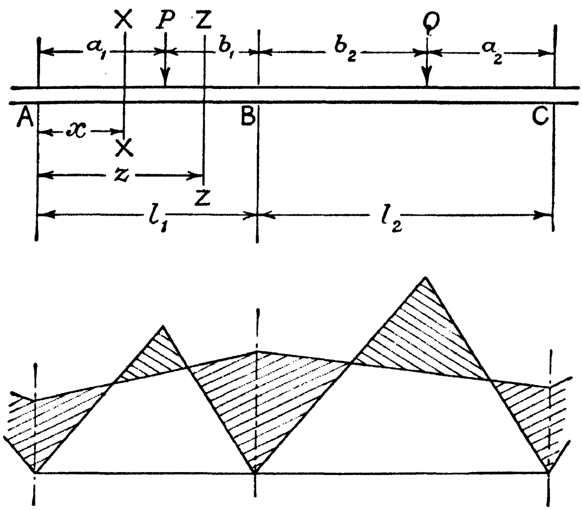


Fig. 4.4

Taking moments about B,

$$R_1 l_1 - P b_1 - M_b + M_a = 0.$$

$$\therefore R_1 = \frac{P b_1 + M_b - M_a}{l_1}.$$

The bending moment at A $= M_a$, whilst the change in the bending moment $= R_1 x$.

Hence the bending moment at XX $= R_1 x + M_a$.

$$EI \frac{d^2y}{dx^2} = R_1 x + M_a.$$

$$EI \frac{dy}{dx} = \tfrac{1}{2} R_1 x^2 + M_a x + A_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$EIy = \tfrac{1}{6} R_1 x^3 + \tfrac{1}{2} M_a x^2 + A_1 x + B_1. \quad . \quad . \quad . \quad (2)$$

Known conditions: $y = 0$ when $x = 0$, $\therefore B_1 = 0$.

Bending moment at $ZZ = R_1 z + M_a - P(z - a_1)$,

i.e. $$EI\frac{d^2y}{dz^2} = R_1 z + M_a - P(z - a_1).$$

$$\therefore EI\frac{dy}{dz} = \tfrac{1}{2}R_1 z^2 + M_a z - \tfrac{1}{2}Pz^2 + Pa_1 z + A_2, \quad . \quad . \quad . \quad (3)$$

and $$EIy = \tfrac{1}{6}R_1 z^3 + \tfrac{1}{2}M_a z^2 - \tfrac{1}{6}Pz^3 + \tfrac{1}{2}Pa_1 z^2 + A_2 z + B_2. \quad (4)$$

Known conditions: $y = 0$ when $z = l_1$.

$$\therefore 0 = \tfrac{1}{6}R_1 l_1^3 + \tfrac{1}{2}M_a l_1^2 - \tfrac{1}{6}Pl_1^3 + \tfrac{1}{2}Pa_1 l_1^2 + A_2 l_1 + B_2.$$

$$\therefore B_2 = -\tfrac{1}{6}R_1 l_1^3 - \tfrac{1}{2}M_a l_1^2 + \tfrac{1}{6}Pl_1^3 - \tfrac{1}{2}Pa_1 l_1^2 - A_2 l_1.$$

The slope under the load is the same whether obtained from equation (1) or (3). Similarly the deflection is the same whether obtained from (2) or (4). Equating equations (1) and (3) for the conditions under the load, i.e. when $x = a_1$ and $z = a_1$, we have

$$\tfrac{1}{2}R_1 a_1^2 + M_a a_1 + A_1 = \tfrac{1}{2}R_1 a_1^2 + M_a a_1 - \tfrac{1}{2}Pa_1^2 + Pa_1^2 + A_2.$$

$$\therefore A_1 = \tfrac{1}{2}Pa_1^2 + A_2.$$

Equating (2) and (4) for the condition under the load, i.e. when $x = a_1$ and $z = a_1$,

$$\tfrac{1}{6}R_1 a_1^3 + \tfrac{1}{2}M_a a_1^2 + A_1 a_1$$
$$= \tfrac{1}{6}R_1 a_1^3 + \tfrac{1}{2}M_a a_1^2 - \tfrac{1}{6}Pa_1^3 + \tfrac{1}{2}Pa_1^3 + A_1 a_1 - \tfrac{1}{2}Pa_1^3 + B_2.$$

$$\therefore B_2 = \tfrac{1}{6}Pa_1^3.$$

But $B_2 = -\tfrac{1}{6}R_1 l_1^3 - \tfrac{1}{2}M_a l_1^2 + \tfrac{1}{6}Pl_1^3 - \tfrac{1}{2}Pa_1 l_1^2 - A_2 l_1 = \tfrac{1}{6}Pa_1^3$.

$$\therefore A_2 = -\tfrac{1}{6}R_1 l_1^2 - \tfrac{1}{2}M_a l_1 + \tfrac{1}{6}Pl_1^2 - \tfrac{1}{2}Pa_1 l_1 - \frac{Pa_1^3}{6l_1}.$$

When $z = l_1$, the slope

$$\frac{dy}{dz} = \tfrac{1}{2}R_1 l_1^2 + M_a l_1 - \tfrac{1}{2}Pl_1^2$$
$$+ Pa_1 l_1 - \tfrac{1}{6}R_1 l_1^2 - \tfrac{1}{2}M_a l_1 + \tfrac{1}{6}Pl_1^2 - \tfrac{1}{2}Pa_1 l_1 - \frac{Pa_1^3}{6l_1}$$
$$= \tfrac{1}{3}R_1 l_1^2 + \tfrac{1}{2}M_a l_1 - \tfrac{1}{3}Pl_1^2 + \tfrac{1}{2}Pa_1 l_1 - \frac{Pa_1^3}{6l_1}. \quad . \quad . \quad . \quad (5)$$

If we take the span BC we get the slope at the support B,

$$\tfrac{1}{3}R_3 l_2^2 + \tfrac{1}{2}M_c l_2 - \tfrac{1}{3}Ql_2^2 + \tfrac{1}{2}Qa_2 l_2 - \frac{Qa_2^3}{6l_2}. \quad . \quad . \quad (6)$$

But the slope given by equations (5) and (6) will be numerically equal but of opposite sign due to the distances being measured in the reverse direction.

$$\text{Hence}\quad \tfrac{1}{3}R_1l_1^2 + \tfrac{1}{2}M_al_1 - \tfrac{1}{3}Pl_1^2 + \tfrac{1}{2}Pa_1l_1 - \frac{Pa_1^3}{6l_1}$$
$$= -\tfrac{1}{3}R_3l_2^2 - \tfrac{1}{2}M_cl_2 + \tfrac{1}{3}Ql_2^2 - \tfrac{1}{2}Qa_2l_2 + \frac{Qa_2^3}{6l_2}.$$

Substituting the values for R_1 and R_3, i.e.

$$R_1 = \frac{Pb_1 - M_a + M_b}{l_1}$$

and

$$R_3 = \frac{Qb_2 - M_c + M_b}{l_2},$$

$$\frac{(Pb_1 - M_a + M_b)l_1^2}{3l_1} + \tfrac{1}{2}M_al_1 - \tfrac{1}{3}Pl_1^2 + \tfrac{1}{2}Pa_1l_1 - \frac{Pa_1^3}{6l_1}$$
$$= -\frac{(Qb_2 - M_c + M_b)l_2^2}{3l_2} - \tfrac{1}{2}M_cl_2 + \tfrac{1}{3}Ql_2^2 - \tfrac{1}{2}Qa_2l_2 + \frac{Qa_2^3}{6l_2},$$

i.e. $\tfrac{1}{3}Pb_1l_1 - \tfrac{1}{3}M_al_1 + \tfrac{1}{3}M_bl_1 + \tfrac{1}{2}M_al_1 - \tfrac{1}{3}Pl_1^2 + \tfrac{1}{2}Pa_1l_1 - \dfrac{Pa_1^3}{6l_1}$

$$= -\tfrac{1}{3}Qb_2l_2 + \tfrac{1}{3}M_cl_2 - \tfrac{1}{3}M_bl_2 - \tfrac{1}{2}M_cl_2 + \tfrac{1}{3}Ql_2^2 - \tfrac{1}{2}Qa_2l_2 + \frac{Qa_2^3}{6l_2}.$$

$$\therefore\ +\tfrac{1}{6}M_al_1 + \tfrac{1}{3}M_bl_1 + \tfrac{1}{3}M_bl_2 + \tfrac{1}{6}M_cl_2$$
$$= \tfrac{1}{3}Pl_1^2 + \frac{Pa_1^3}{6l_1} - \tfrac{1}{2}Pa_1l_1 - \tfrac{1}{3}Pb_1l_1 + \tfrac{1}{3}Ql_2^2$$
$$+ \frac{Qa_2^3}{6l_2} - \tfrac{1}{2}Qa_2l_2 - \tfrac{1}{3}Qb_2l_2.$$

Substituting $(a_1 + b_1)$ and $(a_2 + b_2)$ for l_1 and l_2 respectively on the right-hand side of the equation,

$$+\tfrac{1}{6}M_al_1 + \tfrac{1}{3}M_bl_1 + \tfrac{1}{3}M_bl_2 + \tfrac{1}{6}M_cl_2$$
$$= \tfrac{1}{3}P(a_1 + b_1)^2 + \frac{Pa_1^3}{6(a_1 + b_1)} - \tfrac{1}{2}Pa_1(a_1 + b_1) - \tfrac{1}{3}Pb_1(a_1 + b_1)$$
$$+ \tfrac{1}{3}Q(a_2 + b_2)^2 + \frac{Qa_2^3}{6(a_2 + b_2)} - \tfrac{1}{2}Qa_2(a_2 + b_2) - \tfrac{1}{3}Qb_2(a_2 + b_2)$$
$$= -\tfrac{1}{6}Pa_1^2 - \tfrac{1}{6}Pa_1b_1 + \frac{Pa_1^3}{6(a_1 + b_1)} - \tfrac{1}{6}Qa_2^2 - \tfrac{1}{6}Qa_2b_2 + \frac{Qa_2^3}{6(a_2 + b_2)}$$
$$= -\frac{1}{6(a_1 + b_1)}(2Pa_1^2b_1 + Pa_1b_1^2) - \frac{1}{6(a_2 + b_2)}(2Qa_2^2b_2 + Qa_2b_2^2)$$
$$= -\frac{Pa_1b_1}{6l_1}(2a_1 + b_1) - \frac{Qa_2b_2}{6l_2}(2a_2 + b_2),$$

or $$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -\frac{Pa_1b_1}{l_1}(2a_1 + b_1) - \frac{Qa_2b_2}{l_2}(2a_2 + b_2)$$

$$= -\frac{Pa_1}{l_1}(2a_1b_1 + b_1^2) - \frac{Qa_2}{l_2}(2a_2b_2 + b_2^2).$$

But $b_1 = l_1 - a_1$, and $b_2 = l_2 - a_2$,

$$\therefore\ M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -\frac{Pa_1}{l_1}(l_1^2 - a_1^2) - \frac{Qa_2}{l_2}(l_2^2 - b_2^2).$$

Example 2.—An aeroplane tail spar is 15 ft. long and is supported at three points, 15 in., 75 in. and 165 in. from the one end. It has four concentrated loads acting on it of 55 lb. each, one at each end of the spar, one at 60 in. and the other 120 in. from one end of the spar. Find (*a*) the bending moment at each support, (*b*) the pressure exerted on each support.

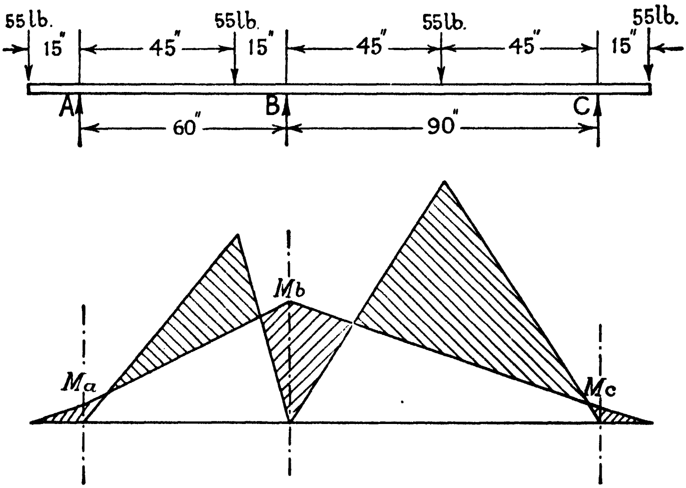


Fig. 4.5

The bending moment at A and C will be due to the load acting on the end of the overhang. Its value will be 55 × 15 = 825 lb.-in. Applying the formula just obtained, namely,

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -\frac{Pa_1}{l_1}(l_1^2 - a_1^2) - \frac{Qa_2}{l_2}(l_2^2 - b_2^2),$$

where $M_a = M_c = -825$ lb.-in., $l_1 = 60$ in., $l_2 = 90$ in., $P = 55$ lb., $a_1 = 45$ in., $Q = 55$ lb., $a_2 = 45$ in., we have

$$-825 \times 60 + 2M_b(60 + 90) - 825 \times 90$$
$$= -\frac{55 \times 45}{60}(60^2 - 45^2) - \frac{55 \times 45}{90}(90^2 - 45^2),$$

i.e. $-49{,}500 + 300M_b - 74{,}250 = -\frac{165}{4} \times 105 \times 15 - \frac{55}{2} \times 135 \times 45$
$$= -64{,}970 - 167{,}100,$$

i.e. $300M_b = -108{,}320,$
$$\therefore M_b = -361{\cdot}1 \text{ lb.-in.}$$

For the reaction at A, take moments about B.

$$R_a \times 60 - 55 \times 75 - 55 \times 15 + 361{\cdot}1 = 0,$$

i.e. $60R_a - 4125 - 825 + 361{\cdot}1 = 0,$

i.e. $60R_a - 4588{\cdot}9 = 0,$
$$\therefore R_a = 76{\cdot}48 \text{ lb.}$$

For the reaction at C, take moments about B.

$$R_c \times 90 - 55 \times 105 - 55 \times 45 + 361{\cdot}1 = 0,$$

i.e. $90R_c - 5775 - 2475 + 361{\cdot}1 = 0.$
$$\therefore 90R_c = 7888{\cdot}9,$$
$$\therefore R_c = 87{\cdot}65 \text{ lb.}$$

Now $R_a + R_b + R_c = \text{total load} = 220 \text{ lb.},$

i.e. $76{\cdot}48 + R_b + 87{\cdot}65 = 220,$
$$\therefore R_b = 55{\cdot}87 \text{ lb.}$$

The reactions could have been found more readily by adding together the shearing forces on each side of each support—for example:

Shearing force to left of A = 55 lb.

$$\text{Shearing force to right of A} = \frac{55 \times 15}{60} + \frac{M_a - M_b}{60}$$
$$= 13{\cdot}75 + \frac{825 - 361{\cdot}1}{60}$$
$$= 13{\cdot}75 + 7{\cdot}73$$
$$= 21{\cdot}48 \text{ lb.}$$
$$\therefore \text{Reaction at A} = 55 + 21{\cdot}48$$
$$= 76{\cdot}48 \text{ lb.}$$

$$\text{Shearing force to left of B} = \frac{55 \times 45}{60} + \frac{M_b - M_a}{60}$$
$$= 41{\cdot}25 - 7{\cdot}73$$
$$= 33{\cdot}52 \text{ lb.}$$

$$\text{Shearing force to right of B} = \frac{55 \times 45}{90} + \frac{M_b - M_c}{90}$$
$$= 27{\cdot}5 + \frac{361{\cdot}1 - 825}{90}$$
$$= 27{\cdot}5 - 5{\cdot}15$$
$$= 22{\cdot}35 \text{ lb.}$$

$$\therefore \text{ Reaction at B} = 33{\cdot}52 + 22{\cdot}35$$
$$= 55{\cdot}87 \text{ lb.}$$

$$\text{Shearing force to left of C} = \frac{55 \times 45}{90} + \frac{M_c - M_b}{90}$$
$$= 27{\cdot}5 + 5{\cdot}15$$
$$= 32{\cdot}65 \text{ lb.}$$

$$\text{Shearing force to right of C} = 55 \text{ lb.}$$

$$\therefore \text{ Reaction at C} = 55 + 32{\cdot}65$$
$$= 87{\cdot}65 \text{ lb.}$$

It is possible to develop Clapeyron's equation by using the area of the bending moment diagram for determining deflections.

Example 3.—A beam is continuous over spans AB and BC, and carries a distributed load of w_1 and w_2 over AB and BC respectively.

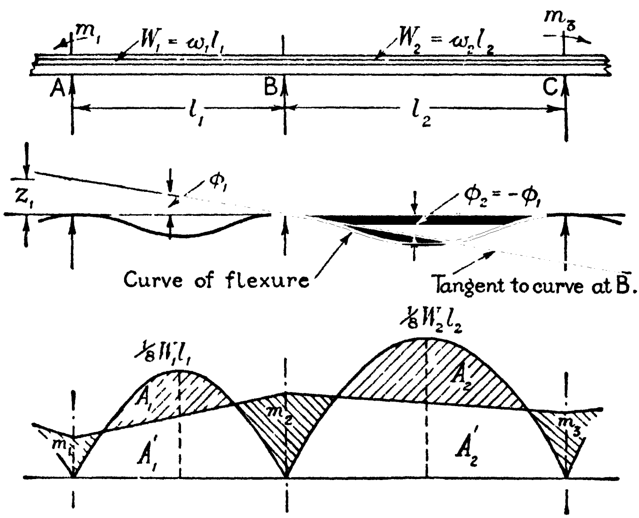


Fig 4.6

$$\phi = \frac{1}{EI}\int M\,dx = \frac{A}{EI}.$$

$$z = \frac{1}{EI}\int Mx\,dx = \frac{A\bar{x}}{EI}.$$

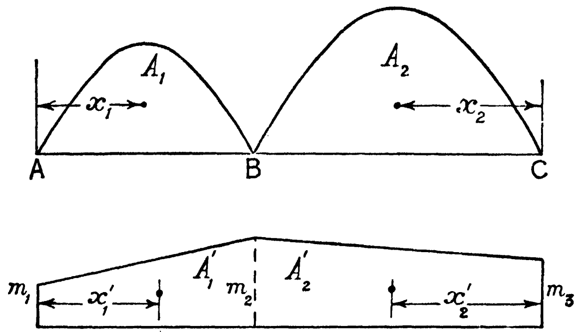


Fig. 4.7.—Bending-moment and fixing-moment diagrams separated

For span AB the deflection at B

$$= z_1 = \frac{A_1 x_1}{EI} + \frac{A_1' x_1'}{EI},$$

or $$Kz_1 = A_1 x_1 + A_1' x_1', \text{ where } K = EI.$$

$$A_1 = \tfrac{2}{3}\cdot\tfrac{1}{8}W_1 l_1 \cdot l_1 = \tfrac{1}{12}W_1 l_1^2.$$

$$x_1 = \tfrac{1}{2}l_1.$$

$$A_1 x_1 = \tfrac{1}{24}W_1 l_1^3, \text{ where } W_1 = w_1 l_1.$$

$$A_1' x_1' = \tfrac{1}{2}m_1 l_1 \times \tfrac{1}{3}l_1 + \tfrac{1}{2}m_2 l_1 \times \tfrac{2}{3}l_1$$

$$= \tfrac{1}{6}(m_1 + 2m_2)l_1^2.$$

$$\therefore\ Kz_1 = \tfrac{1}{24}W_1 l_1^3 + \tfrac{1}{6}(m_1 + 2m_2)l_1^2.$$

But $$z_1 = l_1 \tan\phi_1,$$

$$\therefore\ Kl_1 \tan\phi_1 = \tfrac{1}{24}W_1 l_1^3 + \tfrac{1}{6}(m_1 + 2m_2)l_1^2,$$

or $$K\tan\phi_1 = \tfrac{1}{24}W_1 l_1^2 + \tfrac{1}{6}(m_1 + 2m_2)l_1.$$

Similarly, by treating span CB,

$$K\tan\phi_2 = \tfrac{1}{24}W_2 l_2^2 + \tfrac{1}{6}(m_3 + 2m_2)l_2.$$

But $$\tan\phi_1 = -\tan\phi_2,$$

$$\therefore\ \tfrac{1}{24}W_1 l_1^2 + \tfrac{1}{6}(m_1 + 2m_2)l_1 = -\left\{\tfrac{1}{24}W_2 l_2^2 + \tfrac{1}{6}(m_3 + 2m_2)l_2\right\},$$

or $$(m_1 + 2m_2)l_1 + (m_3 + 2m_2)l_2 = -\tfrac{1}{4}W_1 l_1^2 - \tfrac{1}{4}W_2 l_2^2.$$

$$\therefore\ m_1 l_1 + 2m_2(l_1 + l_2) + m_3 l_2 = -\tfrac{1}{4}\left\{W_1 l_1^2 + W_2 l_2^2\right\}.$$

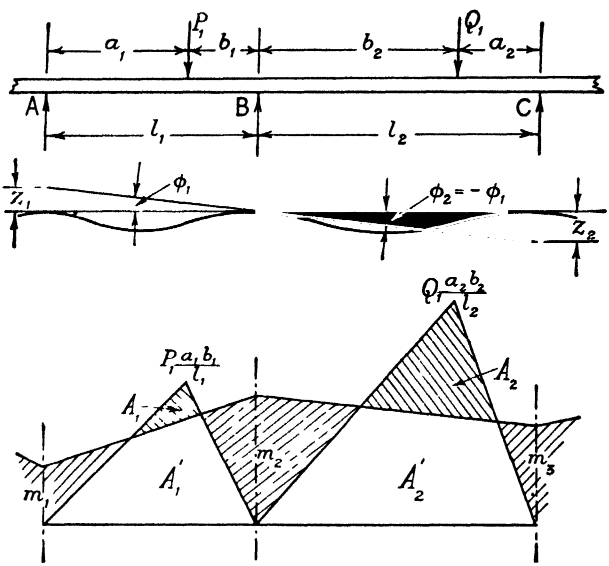


Fig. 4.8

Example 4. A continuous beam with concentrated loads placed as shown in fig. 4.8 may be solved in the following manner.

For span AB,

$$EIz_1 = A_1x_1 + A_1'x_1'.$$

$$A_1 = \tfrac{1}{2}\frac{P_1b_1}{l_1}\cdot a_1l_1 = \tfrac{1}{2}P_1a_1b_1,$$

for the reaction at A due to $P_1 = \dfrac{P_1b_1}{l_1}$.

$$x_1 = \tfrac{1}{2}l_1 + \tfrac{1}{3}(a_1 - \tfrac{1}{2}l_1)$$

$$= \tfrac{1}{3}(l_1 + a_1).$$

$$\therefore\ A_1x_1 = \tfrac{1}{6}P_1a_1b_1(l_1 + a_1),$$

and $A_1'x_1' = \tfrac{1}{6}(m_1 + 2m_2)l_1^2$, as in Example 3.

But $EIz_1 = EIl_1\tan\phi_1 = \tfrac{1}{6}P_1a_1b_1(l_1 + a_1) + \tfrac{1}{6}(m_1 + 2m_2)l_1^2.$

For span CB,

$$EIl_2\tan\phi_2 = EIz_2 = \tfrac{1}{6}Q_1a_2b_2(l_2 + b_2) + \tfrac{1}{6}(m_3 + 2m_2)l_2^2.$$

But $\tan\phi_1 = -\tan\phi_2.$

$$\therefore\ EI\tan\phi_1 = -EI\tan\phi_2.$$

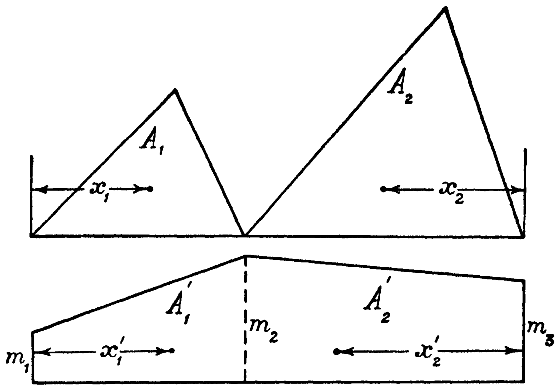


Fig. 19.—Bending-moment and fixing-moment diagrams separated.

$$\therefore\ \tfrac{1}{6}P_1a_1b_1\frac{(l_1+a_1)}{l_1}+\tfrac{1}{6}(m_1+2m_2)l_1=-\left\{\tfrac{1}{6}Q_1a_2b_2\frac{(l_2+b_2)}{l_2}+\tfrac{1}{6}(m_3+2m_2)l_2\right\}.$$

$$\therefore\ (m_1+2m_2)l_1+(m_3+2m_2)l_2=-\frac{P_1a_1b_1(l_1+a_1)}{l_1}-\frac{Q_1a_2b_2(l_2+b_2)}{l_2},$$

or
$$m_1l_1+2m_2(l_1+l_2)+m_3l_2=-\frac{P_1a_1b_1(l_1+a_1)}{l_1}-\frac{Q_1a_2b_2(l_2+b_2)}{l_2}.$$

But $b_1=l_1-a_1$, and $a_2=l_2-b_2$.

$$\therefore\ m_1l_1+2m_2(l_1+l_2)+m_3l_2=-\frac{P_1a_1(l_1^2-a_1^2)}{l_1}-\frac{Q_1b_2(l_2^2-b_2^2)}{l_2}.$$

For a number of loads on each span,

$$m_1l_1+2m_2(l_1+l_2)+m_3l_2=-\left\{\frac{\Sigma Pa(l_1^2-a^2)}{l_1}+\frac{\Sigma Qb(l_2^2-b^2)}{l_2}\right\},$$

where Σ indicates the sum of the separate terms.

(c) *Beam fixed at one end.*

There are two methods by which a problem of this type may be solved.

First method.—Take the support B as origin. The slope of the tangent to the curve of flexure at A is zero, so that

$$EIl\tan\phi=EIz=0.$$

But
$$z=\frac{A_1x_1+A_1'x_1'}{EI}=0.$$

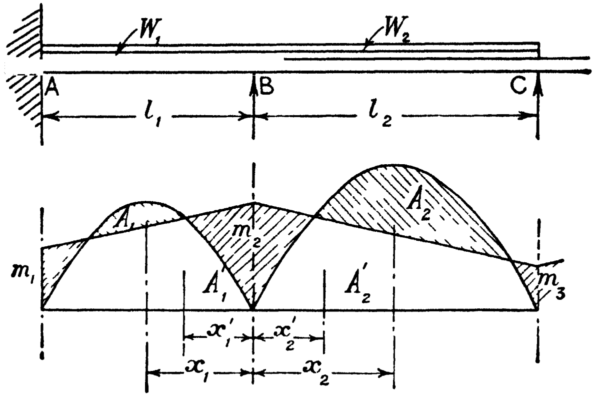


Fig. 4.10

In this problem, where the load is uniformly distributed,

$$A_1 x_1 = \tfrac{1}{24} W_1 l_1^3.$$

$$A_1' x_1' = \left[\tfrac{1}{2}(m_1 + m_2) l_1\right]\left[\tfrac{1}{3} l_1 \left(\frac{m_2 + 2m_1}{m_1 + m_2}\right)\right]$$

$$= \tfrac{1}{6}(2m_1 + m_2) l_1^2.$$

Hence $\qquad EIz = 0 = \tfrac{1}{24} W_1 l_1^3 + \tfrac{1}{6}(2m_1 + m_2) l_1^2,$

$$\therefore\ 2m_1 + m_2 = -\tfrac{1}{4} W_1 l_1.$$

Second method. Imagine an extra span l_0 freely supported at D, the moment at D being m_0.

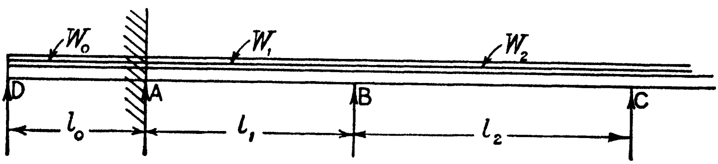


Fig 4 11

From the equation of three moments or Clapeyron's equation,

$$m_0 l_0 + 2m_1(l_0 + l_1) + m_2 l_1 = -\tfrac{1}{4} W_0 l_0^2 - \tfrac{1}{4} W_1 l_1^2.$$

But $l_0 = 0$, $\qquad \therefore\ 2m_1 l_1 + m_2 l_1 = -\tfrac{1}{4} W_1 l_1^2,$

i.e. $\qquad 2m_1 + m_2 = -\tfrac{1}{4} W_1 l_1.$

2. Disadvantage of Continuous Beams.

The disadvantage specified for encastré beams also applies to continuous beams. It is more complicated on account of the difficulty of arranging for all supports to be at the same level.

3. Variation in Level of Supports.

Let support A be distance e below level of other supports.

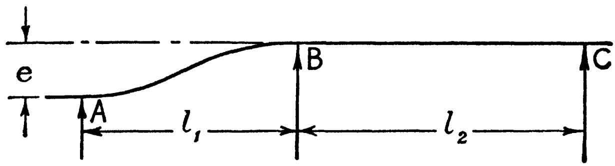


Fig. 4.12

The expression for the deflection of the beam over span AB is given on p. 74 as $EIy = -\frac{1}{24}wx^4 + \frac{1}{6}A_1x^3 + \frac{1}{2}B_1x^2 + C_1x + D_1$.

The conditions will be the same in this problem with the exception that the deflection is e (downwards) when $x = l_1$. Substituting the values obtained for A_1, B_1 and D_1 in the problem on p. 74,

$$EIy = -\tfrac{1}{24}wx_1^4 + \tfrac{1}{12}wl_1x^3 - \frac{M_bx^3}{6l_1} + \frac{M_ax^3}{6l_1} + \tfrac{1}{2}M_bx^2 + C_1x.$$

Now $\qquad y = e$ (downwards when $x = l_1$),

$$\therefore\ EIy = -EIe = \tfrac{1}{24}wl_1^4 + \tfrac{1}{3}M_bl_1^2 + \tfrac{1}{6}M_al_1^2 + C_1l_1.$$

$$\therefore\ C_1 = -\tfrac{1}{24}wl_1^3 - \tfrac{1}{3}M_bl_1 - \tfrac{1}{6}M_al_1 - EIe/l_1.$$

Now $\qquad C_2 = -\tfrac{1}{24}wl_2^3 - \tfrac{1}{3}M_bl_2 - \tfrac{1}{6}M_al_2$ for span BC.

Equating C_1 to $-C_2$,

$$-\tfrac{1}{24}wl_1^3 - \tfrac{1}{3}M_bl_1 - \tfrac{1}{6}M_al_1 - \frac{EIe}{l_1} = +\tfrac{1}{24}wl_2^3 + \tfrac{1}{3}M_bl_2 + \tfrac{1}{6}M_cl_2.$$

$$\therefore\ M_al_1 + 2M_b(l_1 + l_2) + M_cl_2 = -\tfrac{1}{4}wl_1^3 - \tfrac{1}{4}wl_2^3 - \frac{6EIe}{l_1}.$$

If the support B is lower than the other supports by a distance e, then the deflection is e (upwards) when $x = l_1$ and l_2, i.e.

$$EIy = EIe = +\tfrac{1}{24}wl_1^4 + \tfrac{1}{3}M_bl_1^2 + \tfrac{1}{6}M_al_1^2 + C_1l_1.$$

$$\therefore\ C_1 = -\tfrac{1}{24}wl_1^3 - \tfrac{1}{3}M_bl_1 - \tfrac{1}{6}M_al_1 + \frac{EIe}{l_1},$$

and

$$C_2 = -\tfrac{1}{24}wl_2^3 - \tfrac{1}{3}M_bl_2 - \tfrac{1}{6}M_cl_2 + \frac{EIe}{l_2}.$$

Equating C_1 to $-C_2$,

$$-\tfrac{1}{24}wl_1^3 - \tfrac{1}{3}M_bl_1 - \tfrac{1}{6}M_al_1 + \frac{EIe}{l_1} = \tfrac{1}{24}wl_2^3 + \tfrac{1}{3}M_bl_2 + \tfrac{1}{6}M_cl_2 - \frac{EIe}{l_2}.$$

$$\therefore\ M_al_1 + 2M_b(l_1 + l_2) + M_cl_2 = -\tfrac{1}{4}wl_1^3 - \tfrac{1}{4}wl_2^3 + \frac{6EIe}{l_1} + \frac{6EIe}{l_2}.$$

When the beam is simply supported at A and C and there is no overhang, the bending moments at A and C are zero.

If $l_1 = l_2$, then $\quad 2M_b(2l_1) = -\tfrac{1}{2}wl_1^3 + \dfrac{12EIe}{l_1}.$

$$\therefore\ M_b = -\tfrac{1}{8}wl_1^2 + \frac{3EIe}{l_1^2}.$$

Let R_a = reaction at A, then the bending moment at B is

$$R_al_1 - \tfrac{1}{2}wl_1^2 = -\tfrac{1}{8}wl_1^2 + \frac{3EIe}{l_1^2},$$

$$\therefore\ R_a = \tfrac{3}{8}wl_1 + \frac{3EIe}{l_1^3}.$$

By symmetry, $\qquad R_c = \tfrac{3}{8}wl_1 + \dfrac{3EIe}{l_1^3}.$

Since $\qquad R_a + R_b + R_c = 2wl_1,$

$$R_b = 1\tfrac{1}{4}wl_1 - \frac{6EIe}{l_1^3}.$$

4. Moment-distribution Methods applied to Continous Beams.

This system of analysis of continuous-beam problems overcomes the solution of simultaneous equations, which, in the case of beams covering several spans, becomes laborious. After the initial equations are established, the solution becomes one of pure arithmetic. This method was first introduced by Professor Hardy Cross in a paper presented to the American Society of Civil Engineers in 1930, and may in addition be used in the solution of problems on redundant frames.

Initially it is considered that the continuous beam is composed of a number of separate built-in beams, each beam covering one span and being rigidly fixed at the support. The rigidity must be relaxed in order to obtain a complete solution, and this may be carried out by considering that the beam is rotated at each support taken in turn in order to relieve the fixing moment completely. The rotation of one end generally increases the fixing moment at the other end of the beam; consequently, the freeing moment, or induced moment, is less than the original fixing moment at that end of the beam, and the moment

is said to be distributed. The process is continued until all the beams become free at their ends.

As the continuous beam is split up into a number of built-in beams, the condition for a built-in beam must be known in order to calculate the fixing moment. The fixing moments for a beam carrying a uniformly distributed load W over a span L equals $\frac{1}{12}WL$. In addition it is noted that the ends are to be freed by rotating one end of the beam until the fixing moment at that end is relieved. Hence the condition of a beam fixed at one end and rotated at the other must be investigated.

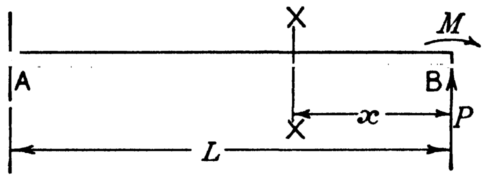


Fig. 4.13

If a beam AB is fixed at A and a clockwise couple together with a vertical force P is applied at B, then the bending moment at section XX is $(Px - M)$.

$$\therefore\ EI\frac{d^2y}{dx^2} = Px - M, \quad \ldots \quad (1)$$

and
$$EI\frac{dy}{dx} = \tfrac{1}{2}Px^2 - Mx + A. \quad \ldots \quad (2)$$

Let $\frac{dy}{dx} = \theta$ when $x = 0$, $\therefore\ A = EI\theta$.

Then
$$EI\frac{dy}{dx} = \tfrac{1}{2}Px^2 - Mx + EI\theta, \quad \ldots \quad (3)$$

and
$$EIy = \tfrac{1}{6}Px^3 - \tfrac{1}{2}Mx^2 + EI\theta x + B. \quad \ldots \quad (4)$$

Now $y = 0$ when $x = 0$, $\quad \therefore\ B = 0$.

The slope equals zero when $x = L$,

$$\therefore\ 0 = \tfrac{1}{2}PL^2 - ML + EI\theta,$$

or
$$EI\theta = ML - \tfrac{1}{2}PL^2. \quad \ldots \quad (5)$$

Also $y = 0$ when $x = L$,

$$\begin{aligned}\therefore\ 0 &= \tfrac{1}{6}PL^3 - \tfrac{1}{2}ML^2 + EI\theta L \\ &= \tfrac{1}{6}PL^3 - \tfrac{1}{2}ML^2 + ML^2 - \tfrac{1}{2}PL^3 \\ &= \tfrac{1}{2}ML^2 - \tfrac{1}{3}PL^3.\end{aligned}$$

$$\therefore\ PL = \tfrac{3}{2}M.$$

Substituting in equation (5) for PL,

$$EI\theta = ML - \tfrac{3}{4}ML$$

$$= \tfrac{1}{4}ML,$$

or

$$M = \frac{4EI\theta}{L}.$$

$$\text{The fixing moment at A} = PL - M$$

$$= \tfrac{3}{2}M - M$$

$$= \tfrac{1}{2}M.$$

$$= \frac{2EI\theta}{L}.$$

Hence if a clockwise moment M be applied at one end of a beam which remains at the same level, then a clockwise moment $\frac{1}{2}M$ acts at the other end if it is to remain fixed.

It will now be possible to calculate the couple which is required to free the ends of the beam from their assumed fixing.

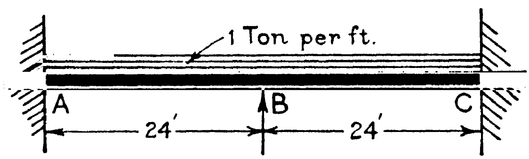


Fig. 4.14

Example 5.—A beam is continuous over a span of 48 ft., but is fixed at both ends and supported in the centre. It carries a uniformly distributed load of 1 ton per foot run over the entire length. For simplicity take the product of the modulus of elasticity and the second moment of the area as 100 tons-ft.^{2} units.

The reaction for separate beams AB, BC will be 12 tons at each end.

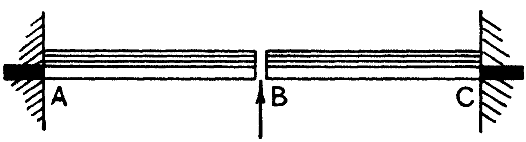


Fig. 4.15

In order to investigate the problem logically it will be best to tabulate the results. It will first be necessary to calculate the value of the rotating moment $4EI\theta/L$ and its component $2EI\theta/L$ for each span.

Span	$\frac{4EI\theta}{L}$	$\frac{2EI\theta}{L}$	Fixing moment, $\frac{1}{12}WL$ tons-ft.	Reaction, tons
AB	$\frac{50}{3}\theta$	$\frac{25}{3}\theta$	A −48 B +48	A 12 B 12
BC	$\frac{50}{3}\theta$	$\frac{25}{3}\theta$	B −48 C +48	B 12 C 12

As both A and C are permanently fixed it will only be necessary to rotate the beam at B in order to obtain the required condition. The moment provided by the fixing at B will be equal to zero, i.e. +48 − 48. Hence no moment distribution is necessary and the fixing moments at A, B and C are each equal to 48 tons-ft., whilst the reactions at A, B and C are 12 tons, 24 tons and 12 tons respectively.

Example 6.- A continuous beam covers 2 spans, one of 24 ft. and the other of 12 ft., and is fixed at the two ends. The load carried is uniformly distributed and of 1 ton per foot run. If the product of the second moment and modulus of elasticity be 100 tons-ft.², find the value of the fixing moments and the reactions at the supports.

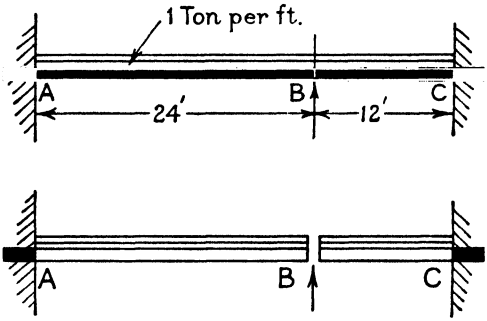


Fig. 4 16

Span	$\frac{4EI\theta}{L}$	$\frac{2EI\theta}{L}$	Fixing moment, $\frac{1}{12}WL$ tons-ft.	Reaction, tons
AB	$\frac{50}{3}\theta$	$\frac{25}{3}\theta$	A −48 B +48	12 12
BC	$\frac{100}{3}\theta$	$\frac{50}{3}\theta$	B −12 C +12	6 6

In order to obtain a balance of the moments at B it will be necessary to rotate the beam in a counter-clockwise direction so that the clockwise moment equal to (48 − 12) tons-ft. is liquidated. Both spans AB and BC must be rotated through the angle θ, so that the total moment required at B, namely, $\frac{1}{3}(50\theta + 100\theta)$, must be applied to AB and BC in the ratio of $\frac{1}{3}(50\theta)$ to $\frac{1}{3}(100\theta)$, i.e. 1 : 2. This extra moment at B will give moments at A in the ratio $\frac{1}{3}(25\theta)$ to $\frac{1}{3}(50\theta)$ of that at B, and at C in the ratio $\frac{1}{3}(50\theta)$ to $\frac{1}{3}(100\theta)$ of that at B. Since the moment to distribute is 36 tons-ft., $\frac{1}{3} \times 36 = 12$ tons-ft. will be applied to AB, and $\frac{1}{3}(2 \times 36) = 24$ tons-ft. to BC.

	M_a	M_b		M_c
Fixed-end moments	−48	+48	−12	+12
Distributed moments	$\frac{25}{150}(48-12)$	$\frac{50}{150}(48-12)$	$-\frac{100}{150}(48-12)$	$-\frac{50}{150}(48-12)$
	−6	−12	−24	−12
Total moment	−54	+36	−36	0

The moments at B are now balanced, so that the net moments at A, B and C are 54 tons-ft., 36 tons-ft. and 0 respectively.

The additional reactions are due to the additional moments. On AB there is an additional counter-clockwise moment of 18 tons-ft., hence the additional reaction will be

$$\pm \frac{18}{\text{span AB}} = \pm\frac{18}{24} = \pm\frac{3}{4} \text{ ton.}$$

The reaction at A will be increased by $\frac{3}{4}$ ton, whilst the reaction at B will be decreased by $\frac{3}{4}$ ton. The additional reaction on BC will be $\pm\frac{36}{12} = \pm 3$ tons. The reaction at B will be increased by 3 tons, whilst that at C will be decreased by 3 tons.

Net reactions are $12\frac{3}{4}$ tons at A,

$18 + 3 - \frac{3}{4} = 20\frac{1}{4}$ tons at B,

$6 - 3 = 3$ tons at C.

We now proceed to solve the above problem by the theorem of three moments. Imagine there is an extra span l_{da} freely supported at D, then

$$m_d l_{da} + 2m_a(l_{ab} + l_{da}) + m_b l_{ab} = -\frac{1}{4}(w_{da} l_{da}^3 + w_{ab} l_{ab}^3).$$

Now $l_{da} = 0$, $\therefore\ 2m_a l_{ab} + m_b l_{ab} = -\frac{1}{4} w_{ab} l_{ab}^3,$

or $2m_a + m_b = -\frac{1}{4} w_{ab} l_{ab}^2$

$= -144.$

Similarly for span BC,

$$2m_c + m_b = -\tfrac{1}{4} \times 1 \times 12^2$$
$$= -36.$$

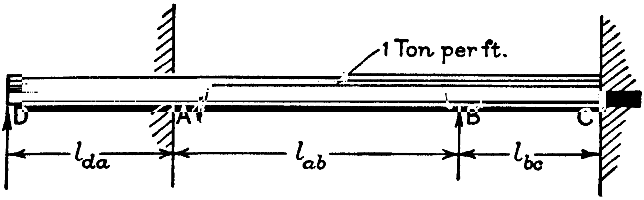


Fig. 4.17

For spans AB and BC,

$$m_a l_{ab} + 2m_b(l_{ab} + l_{bc}) + m_c l_{bc} = -\tfrac{1}{4} w_{ab} l_{ab}^3 - \tfrac{1}{4} w_{bc} l_{bc}^3,$$

i.e.
$$24m_a + 72m_b + 12m_c = -\tfrac{1}{4} \times 1 \times 24^3 - \tfrac{1}{4} \times 1 \times 12^3$$
$$= -3456 - 432$$
$$= -3888.$$

$$\therefore\ 24(-72 - \tfrac{1}{2}m_b) + 72m_b + 12(-18 - \tfrac{1}{2}m_b) = -3888,$$

i.e.
$$-1728 - 12m_b + 72m_b - 216 - 6m_b = -3888.$$

$$\therefore\ 54m_b = -1944,$$

$$\therefore \begin{cases} m_b = -36, \\ m_a = -72 + \frac{36}{2} = -54, \\ m_c = \dfrac{-36 + 36}{2} = 0. \end{cases}$$

Taking moments about B,

$$R_a \times l_{ab} - wl_{ab} \times \tfrac{1}{2}l_{ab} + m_a - m_b = 0.$$
$$\therefore\ R_a \times 24 - 24 \times 12 - 54 + 36 = 0.$$
$$\therefore\ 24R_a = 306.$$
$$\therefore\ R_a = 12\tfrac{3}{4} \text{ tons}.$$

Also
$$R_c \times l_{bc} - wl_{bc} \times \tfrac{1}{2}l_{bc} + m_c - m_b = 0.$$
$$\therefore\ 12R_c - 12 \times 6 + 0 + 36 = 0.$$
$$\therefore\ R_c = 3 \text{ tons}.$$

Now
$$R_a + R_b + R_c = 24 + 12,$$
$$\therefore\ R_b = 36 - 3 - 12\tfrac{3}{4}$$
$$= 20\tfrac{1}{4} \text{ tons}.$$

These results confirm those obtained by the system of moment distribution, but it will be appreciated that the solution takes much longer.

Example 7.—If in the previous example the load on span BC was 2 tons per foot, the solution would have been as shown opposite:

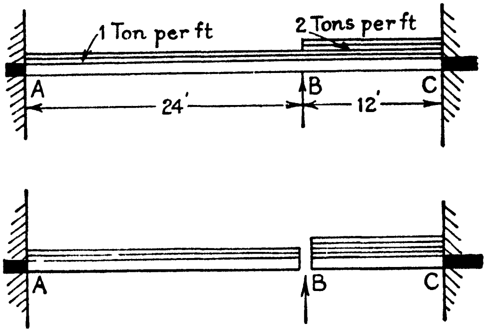


Fig. 4.18

Thus the moments at A, B and C are 52 tons-ft., 40 tons-ft., and 16 tons-ft. respectively. For span AB the extra counter-clockwise moment of 12 tons-ft. will give a change in the reactions of $\frac{12}{24} = \frac{1}{2}$ ton.

$$\therefore \text{ Reaction at A} = 12 + \tfrac{1}{2} = 12\tfrac{1}{2} \text{ tons.}$$

For span BC the extra counter-clockwise moment is 24 tons-ft., hence change in reaction due to this moment is $\frac{24}{12} = 2$ tons.

$$\therefore \text{ Reaction at C} = 12 - 2 = 10 \text{ tons,}$$

and reaction at B $= 24 - \frac{1}{2} + 2 = 25\frac{1}{2}$ tons.

5. Magnitude of rotating couple when beam is simply supported.

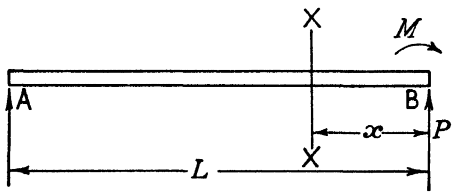


Fig. 4.19

If a beam AB is simply supported at A and the end B rotated, then the equations for bending moment, slope and deflection are

$$EI\frac{d^2y}{dx^2} = Px - M, \quad \ldots \quad (1)$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}Px^2 - Mx + A, \quad \ldots \quad (2)$$

$$EIy = \tfrac{1}{6}Px^3 - \tfrac{1}{2}Mx^2 + Ax + B. \quad \ldots \quad (3)$$

Span	$\frac{4EI\theta}{L}$	$\frac{2EI\theta}{L}$	Fixing moment, $\frac{1}{12}WL$ tons ft.	Reaction, tons
AB	$\frac{50}{3}\theta$	$\frac{25}{3}\theta$	A −48 B +48	12 12
BC	$\frac{100}{3}\theta$	$\frac{50}{3}\theta$	A −24 B - 24	12 12

	M_a	M_b		M_c
Fixed-end moment	−48	- 48	−24	+24
Distributed moment	$\frac{-\frac{1}{3}\times 25\theta}{\frac{1}{3}\times 150\theta}(48-24)$	$\frac{-\frac{1}{3}\times 50\theta}{\frac{1}{3}\times 150\theta}(48-24)$	$\frac{-\frac{1}{3}\times 100\theta}{\frac{1}{3}\times 150\theta}(48-24)$	$\frac{-\frac{1}{3}\times 50\theta}{\frac{1}{3}\times 150\theta}(48-24)$
	−4	−8	−16	−8
Total moment	−52	-40	−40	+16

When $x = 0$, $y = 0$, $\therefore B = 0$;

and $\frac{dy}{dx} = \theta$ when $x = 0$, $\therefore A = EI\theta$;

$$\therefore EIy = \tfrac{1}{6}Px^3 - \tfrac{1}{2}Mx^2 + EI\theta x.$$

When $x = L$, $y = 0$,

$$\therefore EI\theta L = \tfrac{1}{2}ML^2 - \tfrac{1}{6}PL^3.$$

When $x = L$, the bending moment is zero,

$$\therefore PL = M,$$

$$\therefore P = \frac{M}{L}.$$

Hence $$EI\theta L = \tfrac{1}{2}ML^2 - \tfrac{1}{6}ML^2$$

$$= \tfrac{1}{3}ML^2.$$

or $$M = \frac{3EI\theta}{L}.$$

Hence when the rotating couple is equal to $3EI\theta/L$ there is no fixing moment at the other support.

Example 8.—A continuous beam ABC covers two spans, AB 24 ft. long and BC 12 ft. long. A uniformly distributed load of 1 ton per foot covers both spans. The beam is freely supported at A, B and C, which are all at the same level. Find the moment at B and the reactions at A, B and C. The product of the modulus of elasticity and the second moment is 100 tons-ft.^{2} units.

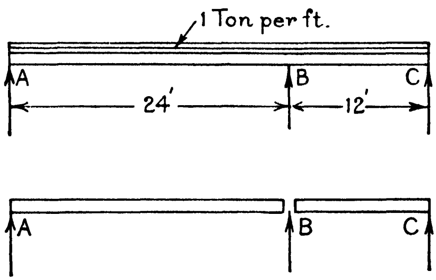


Fig. 4.20

Since the beam is freely supported at A and C, the bending moments at these points are zero. The solution is set out opposite and it is seen that the moments at A, B and C are 0, 54 tons-ft., 0 respectively.

Span	$\frac{4EI\theta}{L}$	$\frac{3EI\theta}{L}$	$\frac{2EI\theta}{L}$	Fixing moment $\frac{1}{12}WL$ tons-ft.	Reaction, tons
AB	$\frac{50}{3}\theta$	$\frac{25}{2}\theta$	$\frac{25}{3}\theta$	A −48 B +48	A 12 B 12
BC	$\frac{100}{3}\theta$	25θ	$\frac{50}{3}\theta$	B −12 C +12	B 6 C 6

	M_a	M_b		M_c
Fixed-end moments	−48	−48	−12	+12
Moment distribution (span AB)	+48	$\frac{25\theta}{3} \cdot \frac{3}{50\theta} \cdot 48 = +24$		
Moment distribution (span BC)			$\frac{50\theta}{3} \cdot \frac{3}{100\theta}(-12) = -6$	−12
Net moments	0	+72	−18	0
Moment distribution (support B)		$-\frac{25\theta}{2} \cdot \frac{2}{75\theta}(72 - 18) = -18$	$-25\theta \cdot \frac{2}{75\theta}(72 - 18) = -36$	
Net moment	0	+54	−54	0

Let the reaction at A be R_a, then, taking moments about B,

$$R_a \times 24 - 24 \times 12 + M_b = 0.$$

$$\therefore R_a = \frac{-54 + 288}{24} = 9\tfrac{3}{4} \text{ tons.}$$

If the reaction at C be R_c, then by moments about B,

$$R_c \times 12 - 12 \times 6 + M_b = 0.$$

$$\therefore R_c = 1\tfrac{1}{2} \text{ tons.}$$

$$R_a + R_b + R_c = \text{total load} = 36 \text{ tons.}$$

$$\therefore R_b = 36 - 9\tfrac{3}{4} - 1\tfrac{1}{2} = 24\tfrac{3}{4} \text{ tons.}$$

Example 9.- A continuous beam, ABC, covers two spans, AB of 24 ft. and BC of 12 ft. A concentrated load of 6 tons is applied at 12 ft. from A, and another concentrated load of 4 tons is applied at 6 ft. from C.

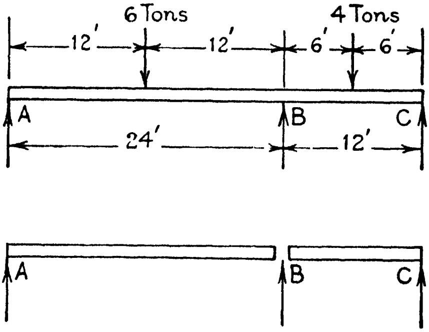


Fig 4 21

If the two ends are freely supported, find the values of the moment at B and the reactions at A, B and C. Let the product of the modulus of elasticity and the second moment of area be 100 tons-ft.² units.

It should be noticed that the fixing moments will be $\frac{1}{8}WL$ since the load is concentrated at the centre of each span. From the tables on p. 99 it is seen that the moments at A, B and C are 0, 21 tons-ft. and 0 respectively.

For the reaction at C, take moments about B, then

$$R_c \times 12 - 4 \times 6 + M_b = 0.$$

$$\therefore R_c = \tfrac{3}{12} = \tfrac{1}{4} \text{ ton.}$$

For the reaction at A, take moments about B, then

$$R_a \times 24 - 6 \times 12 + M_b = 0.$$

$$\therefore R_a = \tfrac{51}{24} = 2\tfrac{1}{8} \text{ tons.}$$

Now $$R_a + R_b + R_c = \text{total load} = 10 \text{ tons.}$$

$$\therefore R_b = 7\tfrac{5}{8} \text{ tons.}$$

Span	$\frac{4EI\theta}{L}$	$\frac{3EI\theta}{L}$	$\frac{2EI\theta}{L}$	Fixing moment, $\frac{1}{8}WL$ tons-ft	Reaction, tons
AB	$\frac{50}{3}\theta$	$\frac{25}{2}\theta$	$\frac{25}{3}\theta$	A −18 B +18	A 3 B 3
BC	$\frac{100}{3}\theta$	25θ	$\frac{50}{3}\theta$	B −6 C +6	A 2 B 2

	M_a	M_b		M_c
Fixed-end moments	−18	+18	−6	+6
Moment distribution (span AB)	−18	$\frac{25\theta}{3} \cdot \frac{3}{50\theta} \cdot 18 = +9$		
Moment distribution (span BC)			$\frac{50\theta}{3} \cdot \frac{3}{100\theta}(-6) = -3$	−6
Net moment	0	+27	−9	0
Moment distribution (support B)		$-\frac{25\theta}{2} \cdot \frac{2}{75\theta}(27 - 9) = -6$	$-\frac{50\theta}{2} \cdot \frac{2}{75\theta}(27 - 9) = -12$	
Net moment	0	+21	−21	0

EXAMPLES

1. A tailplane front spar is supported at three points A, B and C, where AB = BC = 5 ft. The mean upload on the spar is 280 lb. per ft. run. The bending moment at each support (A and C) due to the overhang is 400 lb.-ft. For the spar section, the overall depth = 5 in.; the second moment of area I = 10 in.^{4} units. Assuming the supports remain collinear, calculate the bending moment, bending stress and pressure at B.

[675 lb.-ft.; 2025 lb. per sq. in; 1510 lb.]

2. A horizontal girder, length 75 ft., carries a uniform load, including its own weight, of 2 tons per foot. It has to be raised by three hydraulic rams placed one under each end and one under the middle. The rams are all of the same diameter and are in hydraulic communication with each other. Determine the position and magnitude of the maximum bending moment on the girder, and the deflection of the centre below the level of the ends. The moment of inertia of the cross-section of the girder is 96,000 in.^{4}, and E = 10,000 tons per sq. inch.

[25 ft. from either end; 0·688 in.]

3. A horizontal beam of uniform section is continuous over three supports A, B, C all at the same level. The beam carries uniformly distributed loads W_1, W_2 on spans AB (= l_1) and BC (= l_2) respectively. Show that the bending moments m_1, m_2, m_3 at the supports A, B, C are related by the formulæ:

$$(m_1 + 2m_2)l_1 + (m_3 + 2m_2)l_2 = \tfrac{1}{4}(W_1l_1^2 + W_2l_2^2), \quad 2m_1 + m_2 = \tfrac{1}{4}W_1l_1.$$

4. A horizontal beam ABCD, 60 ft. long, is firmly built in horizontally at A and rests on supports B and C, all supports being at the same level. AB = 20 ft.; BC = 30 ft.; and CD = 10 ft. A uniformly distributed load of $\frac{1}{2}$ ton per ft. run covers the two spans A to C, and an isolated load of 2 tons is placed at the free end of the overhang CD. Determine (i) the bending moments at A, B and C, and (ii) the pressures on the supports at B and C.

[6·8 tons-ft.; 36·4 tons-ft.; 20 tons-ft.; 15·4 tons; 8·95 tons.]

5. A horizontal girder of uniform cross-section rests freely on three supports A, B, C all at the same level. It carries isolated loads on both spans, viz. P_1, P_2, . . . on span AB (l_1) at distances a_1, a_2, . . . from A; and Q_1, Q_2, . . . on span BC (l_2) at distances b_1, b_2, . . . from end C. Show that the bending moment (m) at the support B is given by the formula

$$2m(l_1 + l_2) = -\left\{\frac{\Sigma Pa(l_1^2 - a^2)}{l_1} + \frac{\Sigma Qb(l_2^2 - b^2)}{l_2}\right\}.$$

6. In a continuous beam of two spans AB and BC, the end C is fixed and all the supports are in one plane. If the load is one ton per foot uniform throughout the length, obtain the bending-moment and shear-force diagrams when AB = 15 ft. and BC = 20 ft. State the maximum values of bending moment and shear force.

[34·64 tons-ft; 30·73 tons-ft.; 18·75 tons; 5·45 tons.]

7. A girder of uniform cross-section, 5 ft. 6 in. deep, rests freely on four supports all at the same level. The end spans are each 50 ft. and the central span is 60 ft. A uniformly distributed load of $1\frac{1}{2}$ tons per ft. run (including the weight of the girder) covers all three spans. If the maximum tensile stress in the flanges of the girder is taken at 5 tons per sq. in., find what additional distributed load may with safety be placed on the central span. The girder is free at the ends, and the moment of inertia of the section is 44,000 in.^{4}.

8. A girder of two spans is continuous over three supports A, B, C all at the same level; AB = 80 ft.; BC = 60 ft. The loading is uniformly distributed, being 50 tons on AB and 40 tons on BC. Find the bending moment at B and the pressures on the supports at A, B and C: (i) when the girder rests freely on the end supports at A and C; (ii) when the end A is firmly fixed horizontally.

[(i) 386 tons-ft.; A, 29·8 tons; B, 33·8 tons; C, 26·4 tons.
(ii) 316·7 tons-ft.; A, 24·7 tons; B, 50 tons; C, 25·3 tons.]

CHAPTER V

Stresses in Beams

It has been shown in *Materials and Structures* (p. 38) that for normal beam problems a relationship

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{\rho}$$

holds, where M is the bending moment, I is the second moment of area about the neutral axis, f is the stress on a section of the beam at a distance y from the neutral axis, E is the modulus of elasticity, and ρ is the radius of curvature. In deriving the above equation we neglected any change in shape of the cross-section of the beam, although such a change in shape must obviously take place when the material is stressed. In the case of a cantilever with a vertical load, the upper layers are in tension longitudinally, but in compression laterally; thus the lower layers will be in compression longitudinally and in tension laterally, and a rectangular section ABCD will take up a shape such as $A_1B_1C_1D_1$ (fig. 5.1).

In addition to this change in shape there will be a change brought about by the shearing stresses. It has been shown in *Materials and Structures* under the section " Longitudinal Shear Stress " on p. 69, that the shear stress varies as the distance from the neutral axis changes. For a rectangular section, plane cross-sections before bending become curved after bending, but in addition the longitudinal shear stress will vary across the width of the beam, being zero at the outer edge; hence the plane cross-sections will curve both in the vertical and horizontal directions. Where the bending moment does not vary along the beam there is no longitudinal shear stress and consequently this curvature does not take place. Therefore the theory developed which gave the equation $M/I = f/y = E/\rho$ is only strictly true for beams in which the bending moment is constant throughout its span.

Reference to *Materials and Structures* should be made for the proof of the formula $M/I = f/y = E/\rho$ and for its application to straightforward problems. The effects of combined bending and direct stress as well as the distribution of shear stress are all covered in that book and, with the exception of examples at the end of the present chapter, are not treated here.

In the problems solved in the previous book, the cross-section of the beam has been constant; when it changes, it will be necessary to

incorporate such a change in the equation, if the position and magnitude of the maximum stress is to be obtained.

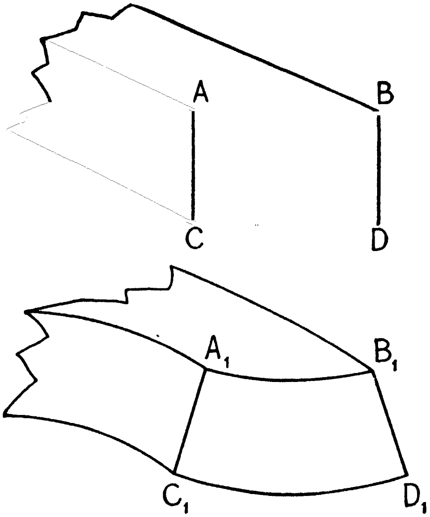


Fig. 5.1

Example 1.—A cantilever, 8 ft. span, of rectangular section is 8 in. broad throughout, but the depth tapers uniformly from 10 in. at the wall to 4 in. at the free end. A load of $\frac{1}{2}$ ton is carried at the free end. Determine the position and magnitude of the maximum stress.

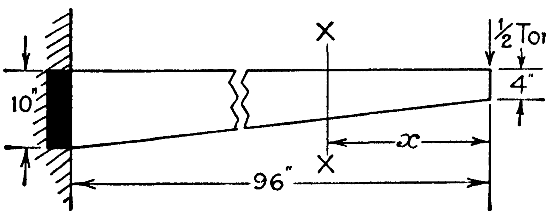


Fig. 5.2

For section XX, $d = (4 + \frac{6}{96}x) = \frac{1}{16}(64 + x)$.

Bending moment at XX $= \frac{1}{2}x$.

Second moment at XX $= \frac{1}{12} \times 8d^3$

$$= \frac{1}{12} \times 8[\tfrac{1}{16}(64 + x)]^3$$

$$= \frac{2}{3 \times 16^3}(64 + x)^3.$$

Now
$$\frac{M}{I} = \frac{f}{y},$$

$$\therefore f = \frac{My}{I} = \frac{\frac{1}{2}x \times \frac{1}{32}(64 + x)}{\frac{2}{3 \times 16^3}(64 + x)^3}$$

$$= \frac{96x}{(64 + x)^2}.$$

The position of maximum stress is where $\frac{df}{dx} = 0$,

i.e.
$$\frac{df}{dx} = 0 = \frac{96(64 + x)^2 - 2(64 + x)96x}{(64 + x)^4}$$

$$= \frac{96(64 + x) - 192x}{(64 + x)^3}.$$

Hence
$$96(64 + x) - 192x = 0,$$

i.e.
$$64 + x - 2x = 0,$$

or
$$x = 64.$$

∴ Position of maximum stress is 64 in. from the free end.

$$\text{Value of maximum stress} = \frac{96 \times 64}{(64 + 64)^2}$$

$$= \frac{96}{4 \times 64}$$

$$= \tfrac{3}{8} \text{ ton per sq. in.}$$

1. Unsymmetrical Bending.

The formula developed for the stresses in beams and used in the earlier part of this chapter was based on the assumption that the transverse section of the beam was symmetrical about an axis passing through the centroid of the section, and parallel to the plane of bending. In some cases the method of loading might be inclined to the principal axis of the section; it is then necessary to split up the bending moment into two components acting along the main axis. In this way one corner of the beam section might be in tension due to the moment on one axis, but in compression due to the moment on the other axis; thus the net result is the algebraic sum of the two stresses. Consider an example.

A rectangular section is inclined at an angle θ whilst the load is applied vertically as shown in fig. 5.3. Let W be the load. Then components of W along OX and OY are $W \sin \theta$ and $W \cos \theta$ respectively. If the beam is simply supported at its ends over a span L with the

concentrated load W at the centre, then the maximum bending moments along axes OX and OY are $W \sin\theta \, . \, \frac{1}{4}L$ and $W \cos\theta \, . \, \frac{1}{4}L$ respectively.

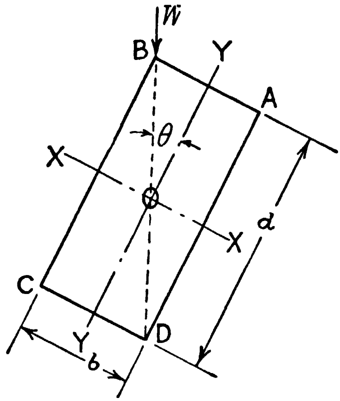


Fig. 5.3

Applying the formula $$\frac{M}{I} = \frac{f}{y},$$

where $I_{YY} = \frac{1}{12}db^3$ about OY,

and $I_{XX} = \frac{1}{12}bd^3$ about OX,

maximum $y = \frac{1}{2}d$ from OX and $\frac{1}{2}b$ from OY.

Then the stresses at A, B, C and D due to the moment $W \cos\theta \, . \, \frac{1}{4}L$ are

$$\frac{W \cos\theta \, . \, \frac{1}{4}L}{\frac{1}{12}bd^3} \, . \, \frac{1}{2}d = f_y.$$

f_y will be tensile for C and D, and compressive for A and B.

The stresses at A, B, C and D due to the moment $W \sin\theta \, . \, \frac{1}{4}L$ are

$$\frac{W \sin\theta \, . \, \frac{1}{4}L}{\frac{1}{12}db^3} \, . \, \frac{1}{2}b = f_x.$$

f_x will be tensile at A and D, and compressive at B and C.

Net stress at D $= f_y + f_x$ tensile.

Net stress at B $= f_y + f_x$ compressive.

Net stress at A $= f_y - f_x$ compressive if positive.

Net stress at C $= f_y - f_x$ tensile if positive.

In the case of angle sections the principal axes are not perpendicular nor parallel to the outer edge of the angle; thus, although the loading might be at right angles to one arm of the angle, the section will be unsymmetrical about the plane of bending. It will therefore be necessary to adopt the foregoing method of solution for such problems.

Example 2.—A British standard unequal angle, 5 in. × 4 in. × ½ in., carries a central load of 1000 lb. as a freely supported beam over a span of 10 ft. The load acts at right angles to the long arm of the angle and through the point of intersection of the principal axes, thus removing any secondary bending. Calculate the stresses at the three outer corners of the angle. The second moments of the area are I_{UU} = 13·111 in.^{4} units and I_{VV} = 3·007 in.^{4} units, and the area of the section is 4·251 sq. in. The other dimensions are as shown in fig. 5·4.

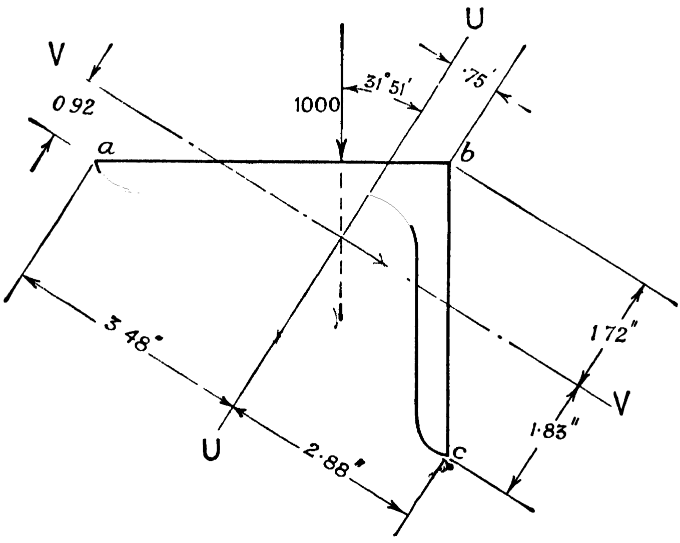


Fig. 5.4

The component of load acting along UU $W \cos 31° \, 51'$

$= 1000 \times \cdot 8495$

$= 849{\cdot}5$ lb.

The component of load acting along VV $= W \sin 31° \, 51'$

$= 1000 \times \cdot 5277$

$= 527{\cdot}7$ lb.

Since the load is concentrated at the centre on a simply supported beam, then the maximum bending moment about UU $= 849 \cdot 5 \times \frac{120}{4}$

$= 25{,}485$ lb.-in.,

and about VV $= 527 \cdot 7 \times \frac{120}{4}$

$= 15{,}830$ lb.-in.

Stress at corner a due to component of load along VV

$$= \frac{15{,}830 \times 3 \cdot 48}{13 \cdot 111}$$

$= 4210$ lb. per sq. in. compressive.

Stress at corner b $= \dfrac{15{,}830 \times \cdot 75}{13 \cdot 111}$

$= 906$ lb. per sq. in. tensile.

Stress at corner c $= \dfrac{15{,}830 \times 2 \cdot 88}{13 \cdot 111}$

$= 3480$ lb. per sq. in. tensile.

Stress at corner a due to component of load along UU

$$= \frac{25{,}485 \times 0 \cdot 92}{3 \cdot 007}$$

$= 7810$ lb. per sq. in. tensile.

Stress at corner b $= \dfrac{25{,}485 \times 1 \cdot 72}{3 \cdot 007}$

$= 14{,}600$ lb. per sq. in. compressive.

Stress at corner c $= \dfrac{25{,}485 \times 1 \cdot 83}{3 \cdot 007}$

$= 15{,}550$ lb. per sq. in. tensile.

Net stress at a $= 7810 - 4210$

$= 3600$ lb. per sq. in. tensile.

Net stress at b $= 14{,}600 - 906$

$= 13{,}694$ lb. per sq. in. compressive.

Net stress at c $= 15{,}550 + 3480$

$= 19{,}030$ lb. per sq. in. tensile.

2. Deflection of Beam subjected to Unsymmetrical Loading.

The deflection of a beam subjected to unsymmetrical loading may be found by resolving the bending moments into the principal planes,

and calculating the deflections caused by these components. The actual deflection is the vector sum of the deflections found from the component moments.

Example 3.—A purlin in a roof truss consists of a joist 4 in. deep and 3 in. wide. The span is 10 ft. and the ends may be considered free. The inclination of the principal rafters of the truss is 30°. If the total vertical load is 1500 lb. and the resultant wind load normal to the side of the roof is 2000 lb., calculate the maximum stress in the purlin and the vertical deflection at its mid-length. The second moment of area of the joist about a line through its centre and parallel to the flanges is 7·5 in.⁴ units, and about a line through its centre and perpendicular to this it is 1·3 in.⁴ units. $E = 30 \times 10^6$ lb. per sq. in. Assume that the load acts through the longitudinal axis of the purlin.

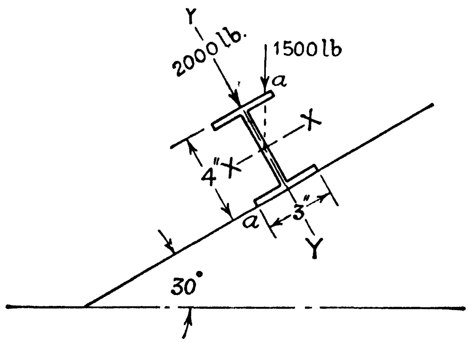


FIG. 5.5

Dead-load bending moment $= \frac{1}{8}WL$

$= \frac{1}{8} \times 1500 \times 10$

$= 1875$ lb.-ft.

$M_{YY} = 1875 \cos 30° = 1875 \times \frac{1}{2}\sqrt{3}$

$= 1622$ lb.-ft.

$M_{XX} = 1875 \sin 30° = 938$ lb.-ft.

Bending moment due to wind load $= \frac{1}{8} \times 2000 \times 10$

$= 2500$ lb.-ft.

Total bending moment acting parallel to YY $= 1622 + 2500$

$= 4122$ lb.-ft.

The stress at corners a will be maximum.

$$\text{Total stress at } a = \frac{938 \times 12 \times 1{\cdot}5}{1{\cdot}3} + \frac{4122 \times 12 \times 2}{7{\cdot}5}$$

$$= 13{,}000 + 13{,}200$$

$$= 26{,}200 \text{ lb. per sq. in.}$$

$$\text{Maximum deflection} = \frac{5}{384}\frac{WL^3}{EI}$$

$$= \frac{1}{8}WL\left(\frac{5L^2}{48EI}\right)$$

$$= (\text{bending moment})\left(\frac{5L^2}{48EI}\right),$$

$$\delta_{YY} = M_{YY} \times \frac{5L^2}{48EI}$$

$$= \frac{4122 \times 5 \times 100 \times 1728}{48 \times 30 \times 10^6 \times 7{\cdot}5}$$

$$= 0{\cdot}33 \text{ in.}$$

$$\delta_{XX} = M_{XX} \times \frac{5L^2}{48EI}$$

$$= \frac{938 \times 5 \times 100 \times 1728}{48 \times 30 \times 10^6 \times 1{\cdot}3}$$

$$= 0{\cdot}432 \text{ in.}$$

$$\text{Net deflection} = \sqrt{(0{\cdot}33^2 + 0{\cdot}432^2)}$$

$$= 0{\cdot}544 \text{ in.}$$

EXAMPLES

1. A horizontal beam of uniform T section is securely built in at its ends, and carries a load of 10 tons uniformly distributed over a span of 20 ft., and a concentrated load of 5 tons at the centre. If the stress due to bending is limited to 6 tons per sq. in., and the deflection to 0·1 in., find the depth of the section. $E = 13{,}000$ tons per sq. in. (London Univ.) [19 in.]

2. A warehouse floor is to be supported by a number of timber joists of square section. The span of the joists is 15 ft. The load on the floor is estimated at 100 lb. per sq. ft. The safe stress in the joists is 1200 lb. per sq. in. and the central deflection is limited to $\frac{1}{2}$ in. Find the necessary cross-sectional dimensions of the joists and their correct distance apart (centre to centre) in feet. $E = 1{\cdot}5 \times 10^6$ lb. per sq. in. [10·8 in.; $7\frac{1}{2}$ ft.]

3. A steel girder of 15 ft. span supports 200 sq. ft. of flooring, carrying a weight of 2 cwt. per sq. ft. Find the dimensions of the girder when

the depth is twice the width over the flanges, and the thickness of metal throughout is $\frac{1}{20}\times$ (flange width). [Depth, 18 in.]

4. A rectangular bar of 4 in. by 2 in. cross-section has a pull of W lb. applied parallel to the axis of the bar, the point of application being $\frac{3}{4}$ in. from one of the longer edges and $1\frac{1}{2}$ in. from one of the shorter edges of the section. If the tensile stress in the material is to be limited to 10 tons per sq. in., determine the value of W and the stress at the four corners of the section. Find the area within which W must be situated if there is to be no compressive stress in the material.

Show your results on a diagram of the cross-section and indicate the position of the neutral axis. (London Univ.)

[27·85 tons; 10 tons per sq. in. tension; 2·61 tons per sq. in. tension; 3·04 tons per sq. in. compression; 4·8 tons per sq. in. tension; middle third.]

5. A masonry column is in section a regular hexagon of 2 ft. side. On a horizontal section at the base the resultant of the external loads is 22·5 tons and acts vertically at a point 4 in. from the centre of the section on a line joining the middle points of opposite sides of the hexagon. The weight of the masonry above the base is 7·5 tons. Find the minimum and maximum stress intensities in tons per sq. ft. on the section. Area of hexagon $A = 2{\cdot}6s^2$; moment of inertia of hexagon about a central diagonal $I = 0{\cdot}54s^4$, where s = length of side.

[3·312 tons per sq. ft. compression; 0·292 ton per sq. ft. compression.]

6. A laminated steel spring, simply supported at the ends and centrally loaded, with a span of 30 in., is required to carry a proof load of 0·75 ton, and the central deflection is not to exceed 2 in. The bending stress must not be greater than 25 tons per sq. in. Plates are available in multiples of $\frac{1}{32}$ in. for thickness and multiples of $\frac{1}{8}$ in. for width. Determine suitable values for the thickness, width and number of plates, and the radius to which the plates should be formed. Assume the width to be twelve times the thickness. (London Univ.)

[$\frac{7}{32}$ in.; $2\frac{5}{8}$ in.; 11; $57\frac{1}{4}$ in.]

7. A leaf spring is built of n parallel steel strips of width b and thickness t. Their centres are clamped together and their lengths are such that each overhangs the one below by an amount a, the overhanging part being triangular. Initially, when the spring is unloaded, the centre lines of each strip are circular arcs of the same radius. The upper strip is of length l equal to $2na$ and when unloaded its central dip is δ. If the spring is subjected to a central force W, show that all the strips will become straight when W reaches the value $\frac{8nEbt^3\delta}{3l^3}$.

8. A compound beam is formed by joining two bars rigidly together, one of steel and the other of brass, each bar being 2 in. wide. The bars are of thickness t_1 and t_2 respectively, so that the total depth is $(t_1 + t_2)$. If Young's modulus for the steel is twice that for the brass, find the ratio $t_1 : t_2$ so that the neutral axis of the section is at the dividing line of the two bars.

If the total depth of the section is to be 1 in., and the stresses in the

steel and brass are not to exceed 16,000 and 6000 lb. per sq. in. respectively, determine the maximum moment of resistance of the beam. (London Univ.) [1 : 1·414; 2345 lb.-in.]

9. Two rectangular bars, one of brass and the other of steel, each $1\frac{1}{2}$ in. by $\frac{3}{8}$ in., are placed together to form a beam $1\frac{1}{2}$ in. wide and $\frac{3}{4}$ in. deep on two supports 30 in. apart. The brass bar is on top of the steel. Determine the maximum central load which can be applied to the beam if the bars are (*a*) separate and can bend independently, (*b*) firmly secured to each other throughout their length. For brass, $E = 12{\cdot}5 \times 10^6$ lb. per sq. in.

Maximum allowable stress in brass = 10,000 lb. per sq. in. and in steel = 15,000 lb. per sq. in. (London Univ.) [562·5 lb.; 1700 lb.]

10. A timber beam 4 in. wide by 8. in. deep is to be reinforced with two steel plates $\frac{1}{4}$ in. thick. Compare the moments of resistance for the same value of the maximum bending stress in the timber when the plates are alternatively (i) 4 in. wide and fixed to the top and bottom surfaces of the beam, (ii) 8 in. deep and fixed to the vertical sides of the beam. E for steel = $20 \times E$ for timber. (London Univ.) [1·36 : 1.]

11. A timber beam 3 in. wide by 6 in. deep is to be reinforced by bonding strips of aluminimum alloy 3 in. wide on to the top and bottom faces of the timber over the whole length of the beam. If the moment of resistance of the composite beam is to be four times that of the timber beam alone, and the same value of the maximum bending stress in the timber is reached in both cases, determine the thickness of the alloy strip and the ratio of the maximum bending stresses in the alloy strip and timber. E for alloy strip = $7{\cdot}15 \times E$ for timber. (London Univ.) [0·37 in.; 8·04 : 1.]

12. A composite beam is made by bonding duralumin sheets to the upper and lower surfaces of a rectangular wooden spar. The wooden spar is 5 in. deep by 2 in. wide and the duralumin sheets are each 2 in. by 0·05 in. Calculate the greatest load which the beam can carry when loaded as a cantilever if the maximum shearing stress in the wood and bond is limited to 800 lb. per sq. in.

Determine the longest cantilever which can carry a concentrated load of 1500 lb. at its extremity, if the tensile and compressive bending stresses in the duralumin are limited to 50,000 lb. per sq. in., and the bending stresses in the wood are limited to 6000 lb. per sq. in. tensile and 4000 lb. per sq. in. compressive. For duralumin, Young's modulus = 11×10^6 lb. per sq. in. For the wood, Young's modulus = 1×10^6 lb. per sq. in. (London Univ.) [13,300 lb.; 24 in.]

13. Show that, for a point in the web of an I beam which is some distance from the points where the web joins the flanges, the rate of variation of shear stress can be expressed in the form

$$\frac{ds}{dy} = \frac{Fy}{I},$$

where F is the vertical shearing force, I is the moment of inertia of the whole cross-section of the beam about its neutral axis, and y is the distance from the neutral axis to the point being considered.

Hence or otherwise prove that the difference between the maximum and mean shear stresses in the web is given by $\frac{Fd^2}{24I}$, where d is the depth of the web.

14. A B.S.B. is of I section of overall height 8 in. and flange width 5 in. The web thickness is 0·28 in. and the flange thicknesses 0·44 in. The standard taper on the flanges may be neglected and all corners may be assumed sharp. The beam is subjected to transverse loads acting parallel to the web, and at one section the shear force is 10 tons. Determine the maximum vertical shearing stress in the web at this section.

What proportion of the total shear force is carried by the web?

[4·85 tons per sq. in.; 94·5 per cent.]

15. A uniformly tapered bar of circular section in the form of a truncated cone is securely fixed at the larger end; the bar is 30 in. long from the fixing and carries a uniformly distributed load. If the maximum bending stress occurs at a section where the diameter is 3 in. and the distance from the fixing is 10 in., determine the diameters at the fixed and free ends.

If the maximum bending stress is 6 tons per sq. in., determine the greatest bending stress at the fixed end. Neglect the weight of the bar. (London Univ.) [4 in.; 2·36 in.; 5·7 tons per sq. in.]

16. A rolled-steel angle is used as a beam with its flanges in horizontal and vertical planes. The section has the following properties: dimensions 6 in. × 6 in. × 1 in.; centre of area 1·85 in. from either outside edge; principal moments of inertia 55·5 and 14·8 in.^{4} units. Calculate the stresses induced in the three outside corners of the section by a bending moment of 3 tons-ft. acting in a vertical plane through the centre of area of the section. The principal axes of the cross-section are inclined at 45° to the sides. [−·78; −4·74 and +4·5 tons per sq. in.]

17. A 6 in. × 6 in. × $\frac{1}{2}$ in. angle is used as a beam, and is subjected to a bending moment of 30 tons-in. acting in a vertical plane and normal to one side of the section. If the moments of inertia are 31·06 and 7·98 in.^{4} units, and the principal axes are inclined at 45° to the vertical, show that the maximum stress is 7·9 tons per sq. in. The centre of area is 1·66 in. from either outside edge of the section.

CHAPTER VI

Strain Energy

1. Direct Stress-strain Energy.

Some problems on strain energy have been treated in *Materials and Structures,* but it is necessary to extend the work further. Provided that the limit of proportionality is not exceeded and the stress is constant over the cross-section,

$$\begin{aligned}\text{the strain energy} &= \tfrac{1}{2} \times \text{total stress} \times \text{extension}\\ &= \tfrac{1}{2}P \times l\\ &= \tfrac{1}{2}P \times \frac{PL}{AE}\\ &= \tfrac{1}{2}\frac{pA \times pAL}{AE}\\ &= \tfrac{1}{2}\frac{p^2}{E} \times AL\\ &= \tfrac{1}{2}\frac{p^2}{E} \times \text{volume of bar.}\end{aligned}$$

Strain-energy methods are useful for solving some problems, in fact it is the only possible way in some cases. The external work done equals the stress-strain energy, hence by forming the necessary equation a solution is effected.

2. Shear Stress-strain Energy.

Let F be the total shear stress when a shear force produces a deformation x, then shear stress-strain energy $= \tfrac{1}{2}Fx$ if the limit of proportionality has not been exceeded. But

$$\text{shear stress} = \text{shear strain} \times \text{modulus of rigidity.}$$

$$\therefore f_s = \frac{x}{t} \times C \quad \text{(see fig. 6.1).}$$

$$\therefore x = \frac{f_s t}{C},$$

and

$$F = Af_s.$$

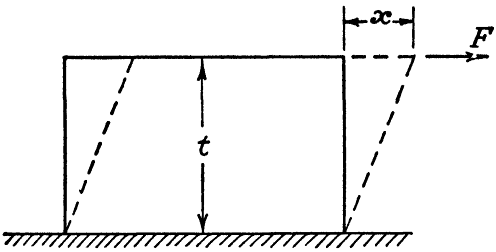


Fig. 6.1

$$\therefore \text{ Shear stress-strain energy} = \tfrac{1}{2}Af_s\frac{f_s t}{C}$$

$$= \tfrac{1}{2}\frac{f_s^2}{C}At$$

$$= \tfrac{1}{2}\frac{f_s^2}{C} \times \text{volume.}$$

3. Torsional Resilience of Circular Shaft.

In this case the shear stress is not constant, so that the manner in which it varies must be taken into account.

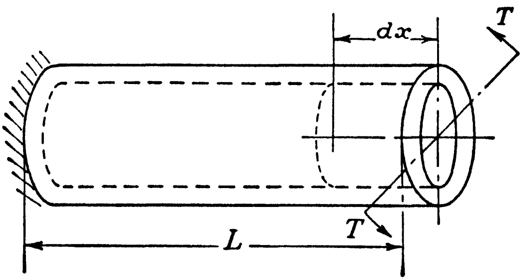


Fig. 6.2

Considering a thin tube of the shaft of length dx, the torsional resilience

$$= \tfrac{1}{2}\frac{f_s^2}{C} \times \text{volume}$$

$$= \tfrac{1}{2}\frac{f_s^2}{C} \times \text{area} \times dx.$$

But
$$\frac{T}{I_0} = \frac{f_s}{r},$$
$$\therefore \quad f_s = \frac{Tr}{I_0}.$$
$$\therefore \text{ torsional resilience} = \tfrac{1}{2} \frac{T^2 r^2}{I_0^2 C} (\text{area})\, dx.$$

The total torsional resilience for the length of shaft dx equals the sum of the torsional resilience for each separate ring making up the shaft and is $\Sigma \frac{1}{2} \frac{T^2 r^2}{I_0^2 C} (\text{area})\, dx$. But T, dx, I_0 and C are constant for this portion of shaft, hence

$$\text{torsional resilience} = \tfrac{1}{2} \frac{T^2\, dx}{I_0^2 C} \Sigma (\text{area}) r^2$$
$$= \tfrac{1}{2} \frac{T^2\, dx I_0}{I_0^2 C}$$
$$= \tfrac{1}{2} \frac{T^2\, dx}{I_0 C}.$$

The torsional resilience for the complete shaft

$$= \int_0^L \tfrac{1}{2} \frac{T^2\, dx}{I_0 C}$$
$$= \tfrac{1}{2} \frac{T^2 L}{I_0 C},$$

and since T is constant for whole shaft,

$$\text{torsional resilience} = \tfrac{1}{2} T \theta,$$

as
$$\theta = \frac{TL}{CI_0}.$$

4. Strain Energy due to Bending.

Strain energy or resilience

$$U = \tfrac{1}{2} \frac{p^2}{E} \times \text{volume of the bar},$$

so that the strain energy per unit volume $= \frac{1}{2} \frac{p^2}{E}$.

In the case of a beam the stress varies from zero at the neutral surface to a maximum at the outer edge of the section.

The strain energy of the small portion of the beam *abcd* of length dx is $\frac{p^2}{2E} \cdot dx \cdot dA$, where dA is the cross-sectional area.

But $$\frac{M}{I} = \frac{p}{y}, \therefore p = \frac{My}{I},$$

and $$\text{strain energy} = \left(\frac{My}{I}\right)^2 \frac{dx \cdot dA}{2E}.$$

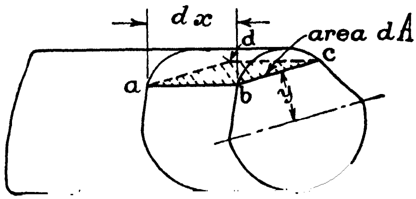


Fig. 6.3

Total strain energy for piece of beam of length dx is

$$\Sigma \frac{M^2 dx}{I^2 \cdot 2E} y^2 dA = \frac{M^2 dx}{2EI^2} \cdot I,$$

where $$I = \Sigma y^2 dA.$$

Total strain energy of beam $$= \int_0^l \frac{M^2 dx}{2EI}$$

$$= \frac{1}{2EI} \int_0^l M^2 dx.$$

Case 1.—*Cantilever with concentrated load W at the free end.*

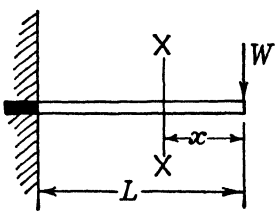


Fig. 6.4

The bending moment $M = Wx$.

$$\text{Strain energy} = \frac{1}{2EI}\int_0^L (Wx)^2\,dx$$

$$= \frac{W^2L^3}{6EI}.$$

Let δ be the deflection at the free end, then the work done by W is $\frac{1}{2}W\delta$ if W is gradually applied.

Equating work done to strain energy,

$$\tfrac{1}{2}W\delta = \frac{W^2L^3}{6EI}. \qquad \therefore\ \delta = \frac{WL^3}{3EI},$$

which is the expression for the maximum deflection obtained by previous methods.

Case 2.—Simply supported beam with concentrated load W at the centre of the span.

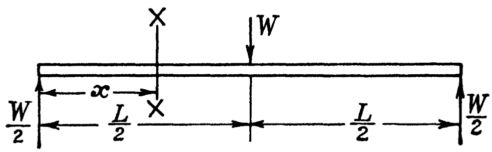


Fig 6 5

As the beam is symmetrically loaded the total strain energy equals twice the strain energy of the beam between the support and the load.

The bending moment at section XX $= \frac{1}{2}Wx$, and this holds between the support and the load.

$$\text{Total strain energy} = 2 \times \frac{1}{2EI}\int_0^{L/2} (\tfrac{1}{2}Wx)^2\,dx$$

$$= \frac{W^2L^3}{96EI}.$$

$$\text{Work done by load} = \tfrac{1}{2}W\delta,$$

$$\therefore\ \delta = \frac{WL^3}{48EI}.$$

Case 3.—Cantilever with a uniformly distributed load w per unit length.

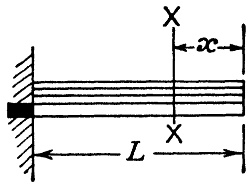


Fig 6.6

The bending moment $= \frac{1}{2}wx^2$.

$$\text{Strain energy} = \frac{1}{2EI}\int_0^L (\tfrac{1}{2}wx^2)^2\,dx$$

$$= \frac{w^2L^5}{40EI} = \frac{W^2L^3}{40EI},$$

where $W = wL$ = total load.

As the total load does not move through the maximum deflection, due to the load being distributed, it is not possible to obtain readily an expression for the work done.

Case 4.—Simply supported beam with a distributed load w per unit length.

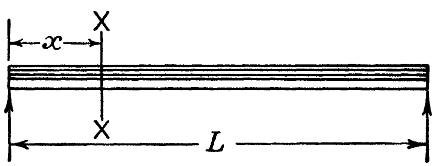


Fig. 6.7

The bending moment $= \frac{1}{2}wLx - \frac{1}{2}wx^2$.

$$\text{Strain energy} = \frac{1}{2EI}\int_0^L (\tfrac{1}{2}wLx - \tfrac{1}{2}wx^2)^2\,dx$$

$$= \frac{w^2L^5}{240EI}$$

$$= \frac{W^2L^3}{240EI},$$

where $W = wL$ = total load.

Case 5.—***Simply supported beam with a concentrated load W at a distance a from left-hand support and b from right-hand support.***

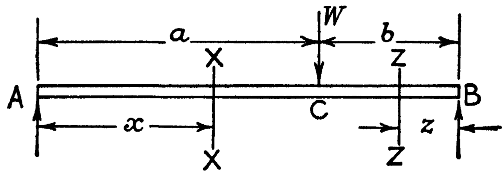


Fig. 6.8

$$\text{Reaction at A} = \frac{Wb}{a+b}.$$

$$\text{Bending moment at XX} = \frac{Wbx}{a+b}.$$

$$\text{Reaction at B} = \frac{Wa}{a+b}.$$

$$\text{Bending moment at ZZ} = \frac{Waz}{a+b}.$$

$$\text{Total strain energy} = \frac{1}{2EI}\int_0^a \left(\frac{Wbx}{a+b}\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Waz}{a+b}\right)^2 dz$$

$$= \frac{1}{2EI}\left[\frac{W^2b^2x^3}{3(a+b)^2}\right]_0^a + \frac{1}{2EI}\left[\frac{W^2a^2z^3}{3(a+b)^2}\right]_0^b$$

$$= \frac{1}{2EI}\left[\frac{W^2b^2a^3}{3(a+b)^2}\right] + \frac{1}{2EI}\left[\frac{W^2a^2b^3}{3(a+b)^2}\right]$$

$$= \frac{W^2a^2b^2}{6EI(a+b)^2}[a+b]$$

$$= \frac{W^2a^2b^2}{6EI(a+b)}.$$

If δ is the deflection under the load, then the work done by W is $\frac{1}{2}W\delta$.

Equating work done to strain energy,

$$\tfrac{1}{2}W\delta = \frac{W^2a^2b^2}{6EI(a+b)},$$

$$\therefore\ \delta = \frac{Wa^2b^2}{3EI(a+b)}.$$

5. Deflection of Beams due to Shear.

In the previous work on deflection of beams the effect of the shearing force on the deflection has been neglected. In order to make the analysis complete it is necessary to study the effect of the shearing forces on the deflection. It will be appreciated from the work done on the distribution of shear stresses that, in the cases where the bending moment is constant over the length of the beam, there is no shearing force and in consequence no deflection due to shear.

It is shown earlier in this chapter that the

$$\text{shear stress-strain energy} = \frac{(\text{shear stress})^2 \times \text{volume}}{2 \times \text{modulus of rigidity}}.$$

In the case of the beam section the shear stress varies as is shown in Chap. III of *Materials and Structures* (p. 69), so that it will be necessary to integrate on the cross-sectional area the elemental shear stress-strain energies in order to obtain the total for the beam. It will therefore be understood that, if the width of the beam section varies, then the problem becomes complex. A solution can be easily obtained for the rectangular beam section and this is the one which will be used in the following examples.

(*a*) *Cantilever with concentrated load at the free end.*

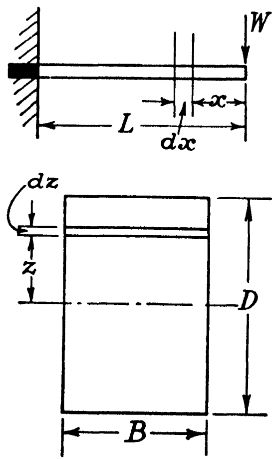


Fig. 6.9

The shear stress at distance z from the neutral axis is

$$f_s = \frac{6F}{BD^3}(\tfrac{1}{4}D^2 - z^2).$$

(See *Materials and Structures*, p. 71, Example 11.)

In this problem the shear force $F = W$, so that

$$f_s = \frac{6W}{BD^3}(\tfrac{1}{4}D^2 - z^2).$$

Let the thickness of the strip be dz and the length of the beam considered dx, then the

$$\text{shear stress-strain energy for the strip} = \frac{f_s^2}{2C} \,.\, dz \,.\, dx \,.\, B,$$

where $dz \times dx \times B$ is the volume.

Total strain energy for piece of beam of length dx

$$= \int_{-\frac{1}{2}D}^{\frac{1}{2}D} \frac{f_s^2}{2C} \,.\, dz \,.\, dx \,.\, B$$

$$= \int_{-\frac{1}{2}D}^{\frac{1}{2}D} \left\{ \frac{6W}{BD^3}(\tfrac{1}{4}D^2 - z^2) \right\}^2 \frac{dz \,.\, dx \,.\, B}{2C}$$

$$= \frac{36W^2}{B^2D^6} \cdot \frac{B\,dx}{2C} \int_{-\frac{1}{2}D}^{\frac{1}{2}D} (\tfrac{1}{16}D^4 - \tfrac{1}{2}D^2z^2 + z^4)\,dz$$

$$= \frac{18W^2\,dx}{BD^6C}\left[\tfrac{1}{16}D^4z - \tfrac{1}{6}D^2z^3 + \tfrac{1}{5}z^5\right]_{-\frac{1}{2}D}^{\frac{1}{2}D}$$

$$= \tfrac{3}{5}\frac{W^2}{BDC}\,dx. \quad \ldots \ldots \quad \text{(i)}$$

Strain energy for whole beam of length L

$$= \tfrac{3}{5}\frac{W^2}{BDC}\int_0^L dx$$

$$= \tfrac{3}{5}\frac{W^2L}{BDC}.$$

If δ_s is the deflection at the free end due to shear, then the work done

$$= \tfrac{1}{2}W\delta_s.$$

Equating work done to strain energy,

$$\tfrac{1}{2}W\delta_s = \tfrac{3}{5}\frac{W^2L}{BDC},$$

$$\therefore\ \delta_s = \tfrac{6}{5}\frac{WL}{BDC}.$$

The total deflection at the free end $= \tfrac{1}{3}\dfrac{WL^3}{EI} + \tfrac{6}{5}\dfrac{WL}{BDC}$

$$= \frac{WL^3}{3EI}\left[1 + \tfrac{18}{5}\frac{IE}{BDCL^2}\right].$$

But I for rectangle $= \frac{1}{12}BD^3$,

$$\therefore \text{ total deflection} = \frac{WL^3}{3EI}\left[1 + \tfrac{3}{10}\frac{E}{C}\cdot\frac{D^2}{L^2}\right].$$

It will be seen that the shear deflection increases in importance as the ratios $E:C$ and $D:L$ increase. Hence it is more likely to be important in deep beams than narrow ones and in wooden beams than steel beams, for the ratio $E:C$ is greater in the case of wood.

(*b*) *Cantilever with uniformly distributed load.*

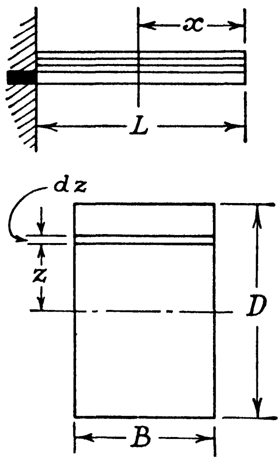


FIG. 6.10

The shearing force F at a section distance x from the free end will be wx, where w is the load per unit length.

$$\text{The shear stress } f_s = \frac{6wx}{BD^3}\left(\tfrac{1}{4}D^2 - z^2\right),$$

and, from (i) above, the total strain energy from piece of beam of length dx

$$= \tfrac{3}{5}\frac{(wx)^2\,dx}{BDC}.$$

Strain energy for whole beam

$$= \tfrac{3}{5}\frac{w^2}{BDC}\int_0^L x^2\,dx$$

$$= \frac{w^2L^3}{5BDC}$$

$$= \frac{W^2L}{5BDC}, \text{ where } W = wL.$$

Equating work done to strain energy,

$$\tfrac{1}{2}W\delta_s = \frac{W^2L}{5BDC},$$

$$\therefore\ \delta_s = \tfrac{2}{5}\frac{WL}{BDC}.$$

$$\text{Total deflection} = \tfrac{1}{8}\frac{WL^3}{EI} + \tfrac{2}{5}\frac{WL}{BDC}.$$

(c) *Simply supported beam with load W concentrated at the centre of span.*

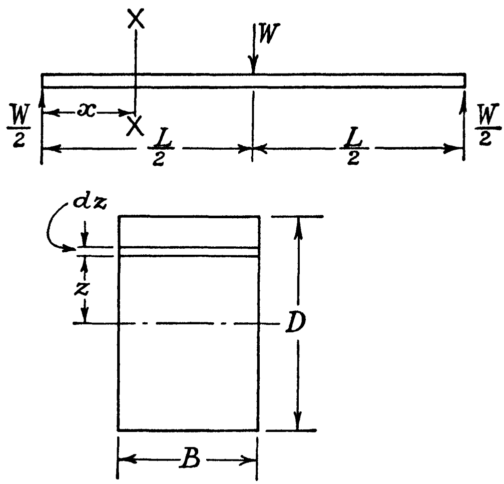


Fig 6.11

The shearing force F between support and load is $\tfrac{1}{2}W$, so that

$$f_s = \frac{3W}{BD^3}\left(\tfrac{1}{4}D^2 - z^2\right),$$

and the total strain energy for a piece of beam of length dx

$$= \int_{-\frac{1}{2}D}^{\frac{1}{2}D}\left\{\frac{3W}{BD^3}\left(\tfrac{1}{4}D^2 - z^2\right)\right\}^2 \frac{dz\,dx\,B}{2C}$$

$$= \frac{9W^2B\,dx}{B^2D^6 2C}\int_{-\frac{1}{2}D}^{\frac{1}{2}D}\left(\tfrac{1}{16}D^4 - \tfrac{1}{2}D^2z^2 + z^4\right)dz$$

$$= \tfrac{3}{20}\frac{W^2}{BDC}\,dx.$$

Total energy for whole beam

$$= \frac{3}{20} \frac{W^2}{BDC} \times 2\int_0^{\frac{1}{2}L} dx$$

$$= \frac{3}{20} \frac{W^2 L}{BDC}$$

Equating work done to total energy, the shear deflection δ_s

$$= \frac{3}{10} \frac{WL}{BDC}$$

$$\therefore \text{ total deflection } = \frac{1}{48} \frac{WL^3}{EI} + \frac{3}{10} \frac{WL}{BDC}$$

(*d*) *Simply supported beam with uniformly distributed load.*

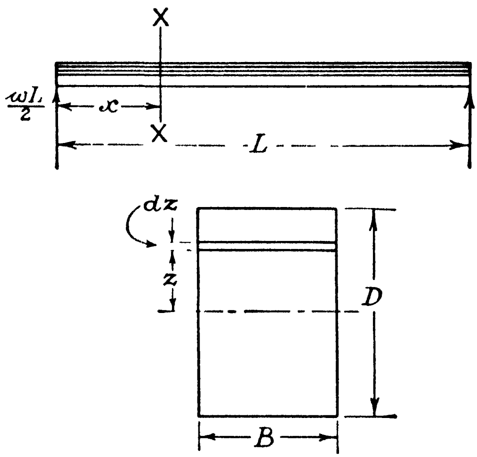


Fig 6.12

The shearing force F at a section distance x from one support

$$= \tfrac{1}{2}wL - wx$$

$$= w(\tfrac{1}{2}L - x).$$

The shear stress $f_s = \frac{6w(\frac{1}{2}L - x)}{BD^3} (\tfrac{1}{4}D^2 - z^2).$

In a certain structure the gradual application of a load W causes a deflection of 0·1 in. at the load-point and a stress of 3 tons per sq. in. in a given member. Find the greatest height from which $\frac{1}{4}W$ can be dropped without exceeding the elastic stress of 16 tons per sq. in. in the member. (London Univ.)

2. A concentrated load w gradually applied to a horizontal beam, simply supported at the ends, produces a deflection y at the load-point. If this load falls through a distance h before making contact with the beam, find an expression, in terms of h and y, for the maximum deflection at the load-point. Neglect loss of energy at impact. In a given beam a concentrated load w, gradually applied, produces a deflection of 0·2 in. at the load-point and a maximum bending-stress of 4 tons per sq. in. Find the greatest height from which a load $0{\cdot}1w$ can be dropped without exceeding the elastic limit stress of 18 tons per sq. in. (London Univ.)

[484 in.]

3. A carriage spring of length L, having n plates of width b and thickness t, carries a central load W. Obtain expressions for (i) the maximum bending stress f in terms of W, L, n, b and t, (ii) the strain energy in terms of f, E and volume of spring.

Determine n and t if $b = 2$ in., $L = 3$ ft. 6 in., $W = 0{\cdot}3$ ton, maximum stress $f = 14$ tons per sq. in. and maximum deflection 2 in. $E = 13{,}000$ tons per sq. in. (London Univ.) [12; 0·237 in.]

4. A laminated-steel carriage spring 4 ft. long consists of 10 leaves $2\frac{1}{2}$ in. wide and $\frac{1}{4}$ in. thick. If the leaves are initially bent to a radius of 7 ft., find the centrally applied load which will just bend the spring flat. What will then be the stress in the material and the resilience of the spring? Neglect friction and take $E = 30 \times 10^6$ lb. per sq. in. (London Univ.)

[21·8 tons.]

5. Compare the strain energy of a beam, simply supported at the ends and loaded with a uniformly distributed load, with that of the same beam centrally loaded and having the same value of the maximum bending stress. (London Univ.)

6. A 10 in. × 6 in. B.S.B., with web 0·4 in. and flanges 0·7 in. thick, has one end built firmly into a wall so that it can act as a horizontal cantilever 12 ft. long. It carries a load of 2 tons placed 6 ft. from the end. Assuming that the shearing force is carried by the web only, the shearing stress being uniformly distributed over the web, calculate the deflection produced at the end of the cantilever. $E = 12{,}800$ tons per sq. in.; $C = 5000$ tons per sq. in. (London Univ.)

7. What additional deflection due to shear expressed as a percentage of the deflection produced by the bending moment occurs at the centre section of an I beam of 12 ft. span, simply supported with a uniformly distributed load of w lb. per ft. run? The beam has a total depth of 24 in., a cross-sectional area of 26·5 sq. in., a thickness of web of 0·5 in., a maximum moment of inertia of 2443 in.^{4}, a breadth of flange of 6·5 in. and a thickness of flange of 1 in. $E = 30 \times 10^6$ lb. per sq. in.; $C = 11{\cdot}54 \times 10^6$ lb. per sq. in. (London Univ.) [2·56.]

8. A metal strip of rectangular cross-section is close coiled to form a flat spiral spring, one end being fixed to a central spindle and the other end to a cylindrical case. Find an expression for the energy stored in the spring due to winding, neglecting that due to shearing stress. Such a spring, made of steel, is required to store 5 ft.-lb. of energy without the normal stress on a cross-section due to bending exceeding 20 tons per sq. in. If the steel weighs 490 lb. per c. ft. and has a Young's modulus equal to 30×10^6 lb. per sq. in., what is the minimum weight of the spring?
[6·1 lb.]

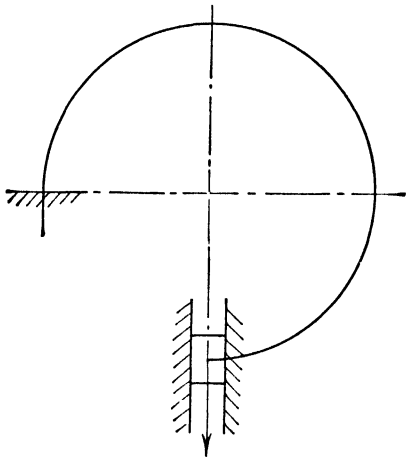


Fig. 6 14

9. Fig. 6.14 shows a steel rod, $\frac{1}{2}$ in. diameter, with one end firmly fixed into a horizontal table. The remainder of the rod is bent into the form of three-quarters of a circle, and the free end of the rod is constrained by guides to move in a vertical direction. If the mean radius to which the rod is bent is 6 in., determine the vertical deflection of the free end when a 50 lb. load is gradually applied there. [0.252 in.]

CHAPTER VII

Torsion

It has been shown in the previous work on torsion that the formula

$$\frac{T}{I_0} = \frac{C\theta}{L} = \frac{f_s}{r}$$

is only applicable for circular shafts. In the case of a square cross-section shaft subjected to torsion there will be a distortion in the shape. If the square section shown in fig. 7.1 is subjected to a twisting moment

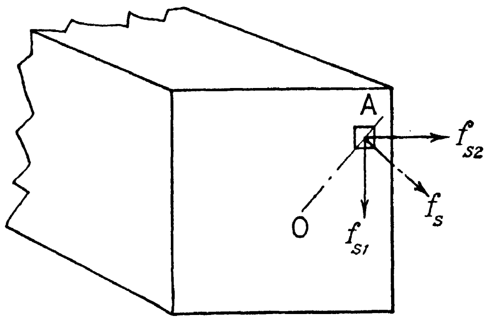


Fig. 7.1

T_1 there will be a shear stress f_s acting on an element at A, where A is near the surface of the shaft. The shear stress will act in a direction at right angles to the radial line OA. The shear stress f_s may be split into two components, one component, f_{s_1}, acting parallel to the side of the square, and the other, f_{s_2}, at right angles to it. It is known from the study of compound stresses that there must be equal shear stresses on planes at right angles to those on which f_{s_1} and f_{s_2} act. The condition is as shown in fig. 7.2.

In the case of a circular shaft there will be only the shear stress f_{s_1} parallel or tangential to the side of the section with its shear stress of equal value at right angles to it. The stresses f_{s_2} will affect both the

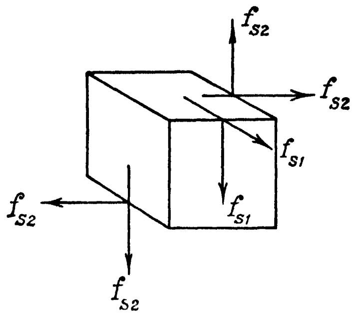


Fig. 7.2

shape and the plane sections of the square shaft. In fig. 7.3 the alteration of the shape is shown in the end view, whilst the plane sections have been warped as shown by the parallel lines in the front elevation.

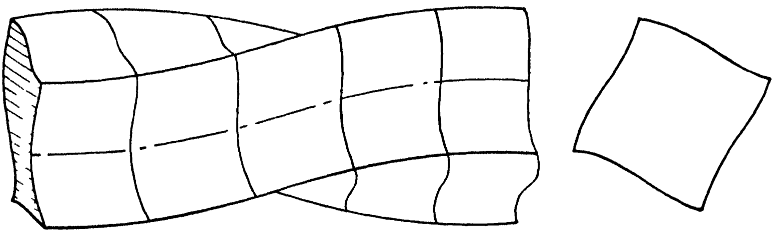


Fig. 7.3

1. Stresses in Rectangular Shaft.

The solution of non-circular members subjected to torsion may be made either analytically or experimentally. The exact analytical solution is difficult, and approximations are made for cases where the exact results are complex (see St. Venant's work in *Mém. savants étrangers*, 14, 1855, or Todhunter and Pearson's account in *History of the Theory of Elasticity*). Bach and Baumann gave an approximate solution for the stresses in a shaft of rectangular cross-section in *Elasticität und Festigkeit*, 8th ed. They assumed that component of the shear stress on the XX axis (f_x) varies parabolically from a maximum at O to zero at the outer edge, whilst it varies directly as the distance from O on the YY axis. The above assumptions are further explained by reference to fig. 7.4. Expressed mathematically,

$$f_x = f_{s_1}\left(1 - \frac{x^2}{a^2}\right)\frac{y}{b},$$

and
$$f_y = f_{s_2}\left(1 - \frac{y^2}{b^2}\right)\frac{x}{a},$$

where f_{s_1} and f_{s_2} are the shear stresses at M and N respectively.

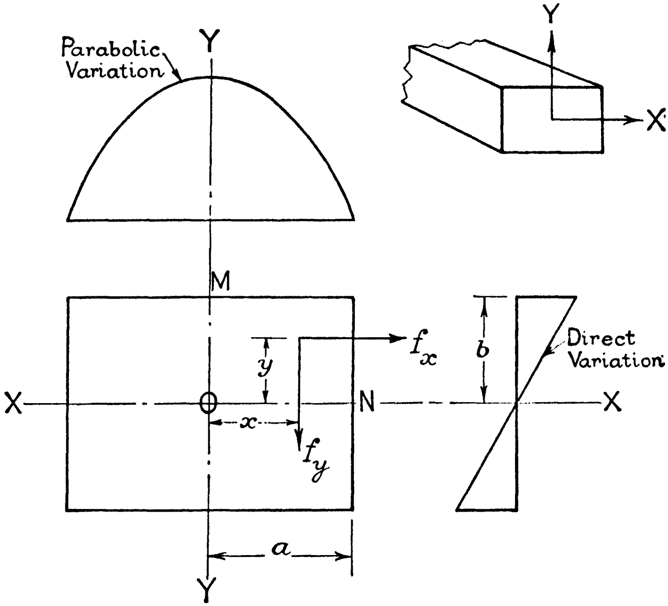


Fig. 7.4

The twisting moment dT to resist the moments of the shear stresses f_{s_1} and f_{s_2} equals $f_x y\,da + f_y x\,da$, where da is the area of the element. The total twisting moment for the shaft

$$T = \int_0^A (f_x y\,da + f_y x\,da)$$

$$= \int_0^A \left\{f_{s_1}\left(1 - \frac{x^2}{a^2}\right)\frac{y}{b}\cdot y\,da + f_{s_2}\left(1 - \frac{y^2}{b^2}\right)\frac{x}{a}\,x\,da\right\}$$

$$= 4\int_0^a\int_0^b \left(f_{s_1}\frac{y^2}{b} - f_{s_1}\frac{x^2y^2}{a^2b} + f_{s_2}\frac{x^2}{a} - f_{s_2}\frac{y^2x^2}{b^2a}\right)dx\,dy,$$

where $da = dx\,dy$.

It is also assumed that f_{s_1} and f_{s_2} are inversely proportional to their distances from the centre of the section, i.e.

$$\frac{f_{s_1}}{a} = \frac{f_{s_2}}{b},$$

or

$$\frac{f_{s_1}}{f_{s_2}} = \frac{a}{b}.$$

Substituting for f_{s_1},

$$T = 4\int_0^a \int_0^b \left(f_{s_2}\frac{a}{b}\frac{y^2}{b} - f_{s_2}\frac{a}{b}\frac{x^2y^2}{a^2b} + f_{s_2}\frac{x^2}{a} - f_{s_2}\frac{y^2x^2}{b^2a}\right) dx\,dy$$

$$= 4f_{s_2}\int_0^a \int_0^b \left(\frac{ay^2}{b^2} - \frac{x^2y^2}{ab^2} + \frac{x^2}{a} - \frac{x^2y^2}{ab^2}\right) dx\,dy$$

$$= 4f_{s_2}\int_0^a \int_0^b \left(\frac{ay^2}{b^2} + \frac{x^2}{a} - \frac{2x^2y^2}{ab^2}\right) dx\,dy$$

$$= 16f_{s_2}\frac{a^2b}{9} \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$= 16f_{s_1}\frac{ab^2}{9}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Therefore

$$f_{s_1} = \frac{9T}{16ab^2},$$

and

$$f_{s_2} = \frac{9T}{16a^2b}.$$

It will be noticed that where $a > b$ the shear stress $f_{s_1} > f_{s_2}$ i.e. the maximum shear stress occurs at M. The results given by these equations may be compared with the exact results given on p. 111 of *Materials and Structures*, where $B = 2b$ and $D = 2a$.

Substituting for a and b in the above equations 1 and 2, we have

$$T = \frac{16}{9}f_{s_2}\,.\,\frac{D^2B}{8}$$

$$= 0{\cdot}22 f_{s_2}\,.\,BD^2$$

$$= k f_{s_2} BD^2,$$

where the value in the exact equation varies from 0·208 to 0·3333, depending on the ratio of B to D.

The angle of twist in a rectangular shaft of length L is found by St. Venant's method to be

$$\theta = \frac{TL}{k_1 BD^3 C},$$

where k_1 is a constant depending on the ratio $B : D$.

Its values are:

B/D	k_1
1	0·141
$1\frac{1}{2}$	0·196
2	0·229
$2\frac{1}{2}$	0·249
3	0·263
∞	0·333

The twisting moment for a circular shaft of diameter D is

$$T = 0{\cdot}196 D^3 f_s,$$

and the angle of twist is $$\theta = \frac{TL}{0{\cdot}098 D^4 C}.$$

It will be noticed that a square shaft of side D has a higher strength and is more rigid than a circular shaft of diameter D.

2. Stresses in Elliptical Shafts.

If the two following assumptions are made, a solution for an elliptical-sectioned shaft may be effected.

(1) The shaft is composed of a number of thin-walled elliptical tubes with no shear stresses at the surface of contact.

(2) The shear-stress components at a point are proportional to their distance from the axis to which they are parallel.

From (1) it follows that the shearing stresses will act tangentially to the boundary of each elliptical tube. Hence the direction of the shearing stresses at any point may be found by differentiating the equation of an ellipse. For example, the equation for the point A_1 in the ellipse shown in fig. 7.5 (p. 136) is

$$\frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} = 1.$$

Therefore the direction of the tangent at A_1 is

$$\frac{dy}{dx} = -\frac{b_1^2 x_1}{a_1^2 y_1}.$$

The relationship between the components of the shearing stresses at A_1 is $\left(-\dfrac{dy}{dx}\right)$, since they act tangentially to the ellipse,

$$\therefore \frac{f_{y_1}}{f_{x_1}} = \frac{b_1^2 x_1}{a_1^2 y_1} \text{ for } A_1, \text{ and } \frac{f_y}{f_x} = \frac{b^2 x}{a^2 y} \text{ for A.}$$

From assumption (2),

$$\frac{f_x}{y} = \frac{f_{x_1}}{y_1} = \text{a constant,}$$

and

$$\frac{f_y}{x} = \frac{f_{y_1}}{x_1} = \text{a constant.}$$

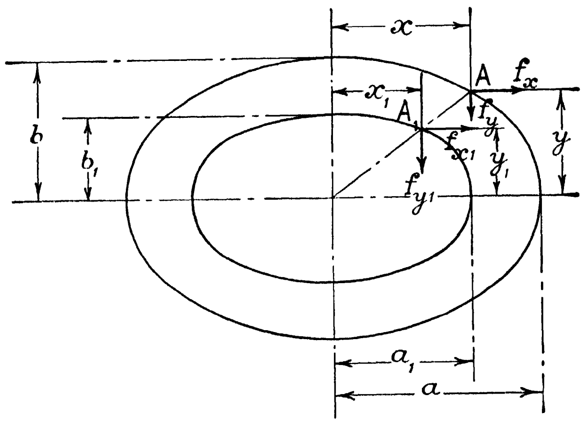


Fig. 7.5

Now the torque is given by

$$dT = (f_{x_1} y_1 + f_{y_1} x_1)\, da,$$

and

$$T = \int_0^A (f_{x_1} y_1 + f_{y_1} x_1)\, da$$

$$= \int_0^A \left(\frac{f_x y_1^2}{y} + \frac{f_y x_1^2}{x}\right) da$$

$$= \frac{f_x}{y} I_x + \frac{f_y}{x} I_y,$$

where I_x and I_y are the second moments of the area with respect to the axes XX and YY respectively.

But
$$\frac{f_x}{y} = \frac{a^2 f_y}{b^2 x}.$$

$$\therefore\ T = \frac{a^2 f_y}{b^2 x} I_x + \frac{f_y}{x} I_y$$
$$= \frac{f_y}{x}\left[\frac{a^2}{b^2} I_x + I_y\right]$$
$$= \frac{f_y}{x}\left[\frac{a^2}{b^2}\frac{\pi a b^3}{4} + \frac{\pi b a^3}{4}\right]$$
$$= \frac{\pi a^3 b}{2x} f_y.$$

Hence
$$f_y = \frac{2Tx}{\pi a^3 b}.$$

When $x = a$, i.e. at the end of the major axis, the shear stress

$$f_{s_1} = \frac{2Ta}{\pi a^3 b} = \frac{16T}{\pi d_2 d_1^2},$$

where d_1 and d_2 are the major and minor axes respectively.

Similarly the shear stress at the end of the minor axis is

$$f_{s_2} = \frac{2T}{\pi a b^2} = \frac{16T}{\pi d_1 d_2^2}.$$

The ratio of the shear stresses at the end of the minor and major axes is

$$\frac{f_{s_2}}{f_{s_1}} = \frac{16Tb}{\pi a b^3} \bigg/ \frac{16Ta}{\pi a^3 b} = \frac{a^2 b}{a b^2} = \frac{a}{b}.$$

Hence the maximum shear stress in an elliptical shaft will occur at the end of the minor axis. The above results are identical with those obtained by the use of St. Venant's theory.

3. Experimental Methods.

The experimental methods used for ascertaining the shear strain or stress conditions in a non-circular shaft are:

(*a*) strain measurement by the use of electrical strain gauges or brittle varnishes painted on the member to be stressed;

(*b*) membrane analogy. When a membrane is stretched across a hole in a plate and is subjected to a uniform surface tension, the equations representing the shape of the membrane are of the same form as those representing the torsional rigidity and stress of a member of uniform cross-section similar to the hole in the plate.

Prandtl drew attention to this fact in 1903, and Griffiths and Taylor devised the soap-film method in 1917.

In order that the results may be of practical use it is always necessary to use a circular hole for calibration purposes, so that two holes are normally cut in the plate, one circular, and the other of the shape under consideration. The plate is fixed to an air tight box, and the films are blown out slightly by a measured quantity of air so that the following measurements may be made:

(i) the inclination of the film surface at the point of contact with the plate;
(ii) the location of a series of contour lines for the deflected membrane or film;
(iii) the volumes of the circular film and the film of the shape under consideration.

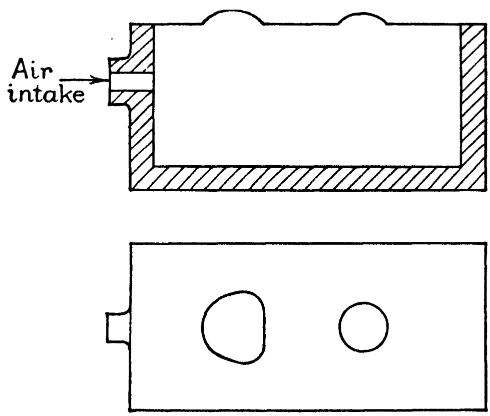


Fig. 7.6

The measurements of the inclination may be made by optical means. The volume of the film for the circular hole may be calculated, since it will be spherical in form, and the deflection of the film at the centre may be measured. Having calculated the volume of the circular film, the volume of the other film may be deduced, since a measured quantity of air is used. A micrometer depth-gauge may be used for mapping the contour lines.

Knowledge of the inclination of the film enables the stress to be calculated, whilst by using the volume measurements the torsional rigidity can be calculated. The contour lines enable a picture of the stress distribution to be obtained. A rapid rise in the contours signifies a concentration of stress, for the stress is proportional to the slope of the film.

4. Shafts of Varying Cross-section.

When the cross-section varies it is necessary to take account of the variation in deducing the polar second moment of area. The same equations will hold as for a shaft of uniform cross-section, but the value of I_0 and r will not be constant.

Example 1.—A shaft, 5 ft. 10 in. long, tapers uniformly from 2 in. diameter at one end to $3\frac{1}{2}$ in. diameter at the other end, where it is firmly held. If a pure axial torque of 1000 lb.-ft. be applied at the 2 in. end, find (i) the maximum shear stress induced in the shaft material, and (ii) the total angle of twist θ in degrees. Take $C = 11 \times 10^6$ lb. per sq. in.

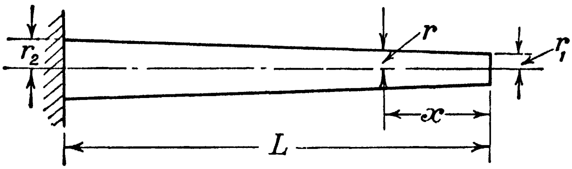


Fig. 7.7

For any section XX, $$\frac{T}{(I_0)_{XX}} = \frac{f_{s_x}}{r_x},$$

where $(I_0)_{XX} = \frac{1}{2}\pi r^4$.

But T is constant for all sections, so that

$$T = f_{s_1} \cdot \frac{\frac{1}{2}\pi r_1^4}{r_1} = f_s \frac{\frac{1}{2}\pi r^4}{r}.$$

$$\therefore f_{s_1} r_1^3 = f_s r^3.$$

The angle of twist θ is given by $TL/(CI_0)$, where T and C are constants. For a small length of shaft dx, where the radius is r, the angle of twist

$$d\theta = \frac{T\,dx}{C \cdot \frac{1}{2}\pi r^4}. \qquad \qquad (1)$$

In this equation there are two variables, x and r, so that it is necessary to connect them by another equation. In the problem under consideration the shaft tapers uniformly, so that

$$r = r_1 + (r_2 - r_1)x/L.$$

Substituting in equation (1) for r,

$$d\theta = \frac{T\,dx}{C \cdot \frac{1}{2}\pi\{r_1 + (r_2 - r_1)x/L\}^4}.$$

Then
$$\theta = \int_0^L \frac{T\,dx}{C \cdot \frac{1}{2}\pi\{r_1 + (r_2 - r_1)x/L\}^4}$$
$$= \frac{2T}{C\pi}\int_0^L \frac{dx}{\{r_1 + (r_2 - r_1)x/L\}^4}$$
$$= \frac{2T}{C\pi}\int_0^L \frac{dx}{(r_1 + kx)^4}, \text{ where } k = (r_2 - r_1)/L.$$

Let
$$u = r_1 + kx,$$
$$\therefore\ du = k\,dx.$$

Hence
$$\theta = \frac{2T}{C\pi}\int \frac{du}{ku^4}$$
$$= \frac{2T}{C\pi k}\left[-\frac{1}{3u^3}\right]$$
$$= \frac{2T}{C\pi k}\left[-\frac{1}{3(r_1 + kx)^3}\right]_0^L$$
$$= -\frac{2T}{3C\pi k}\left[\frac{1}{(r_1 + kL)^3} - \frac{1}{r_1^3}\right]$$
$$= -\frac{2T}{3C\pi\left(\frac{r_2 - r_1}{L}\right)}\left[\frac{1}{\left\{r_1 + \frac{(r_2 - r_1)L}{L}\right\}^3} - \frac{1}{r_1^3}\right]$$
$$= -\frac{2TL}{3C\pi(r_2 - r_1)}\left[\frac{1}{r_2^3} - \frac{1}{r_1^3}\right]$$
$$= -\frac{2TL}{3C\pi(r_2 - r_1)} \cdot \frac{(r_1^3 - r_2^3)}{r_2^3 r_1^3}$$
$$= \frac{2TL}{3C\pi}\left(\frac{r_2^2 + r_1 r_2 + r_2^2}{r_2^3 r_1^3}\right)$$
$$= \frac{2TL \cdot 16}{3C\pi}\left(\frac{d_2^2 + d_1 d_2 + d_2^2}{d_2^3 d_1^3}\right).$$

Substituting values for T, L, d_2, d_1 and C,

$$\theta = \frac{2 \times 1000 \times 12 \times 70 \times 16}{3 \times 11 \times 10^6 \times 3{\cdot}14}\left(\frac{3{\cdot}5^2 + 3{\cdot}5 \times 2 + 2^2}{3{\cdot}5^3 \times 2^3}\right)$$
$$= {\cdot}0177 \text{ radians}$$
$$\equiv 1{\cdot}02^\circ.$$

The maximum stress occurs at the section with the minimum radius.

From
$$\frac{T}{I_0} = \frac{f_s}{r},$$
$$f_s = \frac{Tr}{I_0}$$
$$= \frac{Tr}{\frac{1}{32}\pi d^4}$$
$$= \frac{1000 \times 12 \times 1 \times 32}{\pi \times 2^4}$$
$$= 7640 \text{ lb. per sq. in.}$$

Example 2. *Compound Shaft.*—A gun-metal sleeve is fixed securely to a steel shaft, and the compound shaft is subjected to a torque. If the torque on the sleeve is twice that on the shaft, find the ratio of the external diameter of the sleeve to the diameter of the shaft.

If the limits of shearing stress in the gun-metal and the steel are 3 and 5 tons per sq. in. respectively, find the torque that may be transmitted by the compound tube when the shaft diameter is 2 in. C for steel $= 2\frac{1}{2} \times C$ for gun-metal.

Let T_s = torque on steel,
T_g = torque on gun-metal,
T = total torque,
θ = angle of twist,
C_s = modulus of rigidity for steel,
C_g = modulus of rigidity for gun-metal,
L = length of shaft,
d = diameter of shaft and internal diameter of sleeve,
D = external diameter of sleeve.

Applying formula,
$$\frac{T_s}{\frac{1}{32}\pi d^4} = \frac{C_s\theta}{L}, \quad \ldots\ldots \quad (1)$$
$$\frac{T_g}{\frac{1}{32}\pi(D^4 - d^4)} = \frac{C_g\theta}{L}. \quad \ldots\ldots \quad (2)$$

From (1),
$$\frac{\theta}{L} = \frac{32T_s}{\pi d^4 C_s}.$$

From (2),
$$\frac{\theta}{L} = \frac{32T_g}{\pi(D^4 - d^4)C_g}.$$

Hence
$$\frac{32T_s}{\pi d^4 C_s} = \frac{32T_g}{\pi(D^4 - d^4)C_g},$$
$$\therefore \frac{T_s}{d^4 C_s} = \frac{T_g}{(D^4 - d^4)C_g}.$$

But $$T_g = 2T_s,$$

and $$C_g = C_s \div 2\tfrac{1}{2},$$

$$\therefore \frac{T_s}{d^4 C_s} = \frac{2 \times T_s \times 2\frac{1}{2}}{(D^4 - d^4)C_s}.$$

$$\therefore \frac{1}{d^4} = \frac{5}{D^4 - d^4},$$

or $$D^4 - d^4 = 5d^4.$$

$$\therefore D^4 = 6d^4,$$

$$\therefore D = 6^{1/4}d.$$

Also $$\frac{T_g}{\frac{1}{32}\pi(D^4 - d^4)} = \frac{f_g}{\frac{1}{2}D},$$

where f_g is the maximum shear stress in the sleeve.

$$\therefore T_g = \frac{\pi f_g(D^4 - d^4)}{16D}$$

$$= \frac{\pi \times 3(6d^4 - d^4)}{16 \times 6^{1/4}d}$$

$$= \frac{\pi \times 3 \times 5 \times 2^4}{16 \times 6^{1/4} \times 2}$$

$$= 15{\cdot}1 \text{ tons-in.}$$

Similarly, $$\frac{T_s}{\frac{1}{32}\pi d^4} = \frac{f_s}{\frac{1}{2}d},$$

$$\therefore T_s = \tfrac{1}{16}\pi \times f_s \times d^3$$

$$= \tfrac{1}{16}\pi \times 5 \times 2^3$$

$$= 7{\cdot}85 \text{ tons-in.}$$

Total torque $T = 15{\cdot}1 + 7{\cdot}85$

$= 22{\cdot}95$ tons-in.

5. Open-coiled Helical Springs.

In the case of a close-coiled spring subjected to an axial load it was shown in *Materials and Structures*, Chap. V, that both a twisting moment and a bending moment were acting. As the inclination of the coils was small, the twisting moment $WR \cos \alpha$ was taken as WR, whilst the bending moment $WR \sin \alpha$ was neglected. With open-coiled springs the values of both $\cos \alpha$ and $\sin \alpha$ will affect the result, and therefore cannot be neglected. The twisting moment $WR \cos \alpha$ will act about the axis of the wire, whilst the bending moment will act at right angles to it.

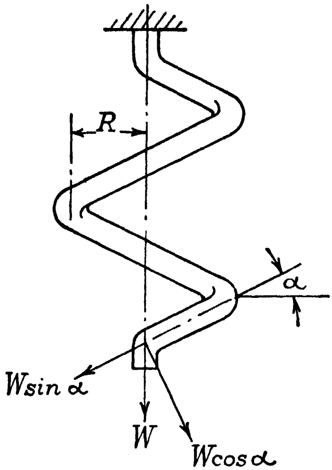


Fig. 7.8

$$\text{Work done by } W = \tfrac{1}{2}W\delta, \text{ where } \delta = \text{extension.}$$

$$\text{Stress-strain energy} = \tfrac{1}{2}WR\cos\alpha\,\frac{TL}{I_0C} + \tfrac{1}{2}WR\sin\alpha\,\frac{ML}{IE},$$

where the twisting moment T and the bending moment M are constant, and I_0 and I represent the polar second moment of area and the second moment about the neutral surface respectively.

$$\text{Hence}\quad \tfrac{1}{2}W\delta = \tfrac{1}{2}WR\cos\alpha\,\frac{TL}{I_0C} + \tfrac{1}{2}WR\sin\alpha\,\frac{ML}{IE}$$

$$= \tfrac{1}{2}(WR\cos\alpha)^2\,\frac{L}{I_0C} + \tfrac{1}{2}(WR\sin\alpha)^2\,\frac{L}{IE}.$$

$$\therefore\ \delta = \frac{WR^2L\cos^2\alpha}{I_0C} + \frac{WR^2L\sin^2\alpha}{IE}.$$

The length of each coil $= 2\pi R\sec\alpha$,

and the length of n coils $= L = 2\pi Rn\sec\alpha$.

$$\therefore\ \delta = \left\{\frac{WR^2\cos^2\alpha}{I_0C} + \frac{WR^2\sin^2\alpha}{IE}\right\}2\pi Rn\sec\alpha$$

$$= \frac{WR^2 2\pi Rn\sec\alpha}{\frac{1}{32}\pi d^4}\left\{\frac{\cos^2\alpha}{C} + \frac{2\sin^2\alpha}{E}\right\},$$

where $I_0 = 2I = \frac{1}{32}\pi d^4$,

$$= \frac{64WR^3n\sec\alpha}{d^4}\left\{\frac{\cos^2\alpha}{C} + \frac{2\sin^2\alpha}{E}\right\}.$$

6. Close-coiled Spring with Axial Torque.

The axial torque M may be split into two components: (*a*) normal to the axis of the coil, $M \cos \alpha$, (*b*) parallel to the axis of the coil, $M \sin \alpha$. The component $M \cos \alpha$ will give a constant bending moment, whilst the component $M \sin \alpha$ gives a constant twisting moment. As the inclination of the coils is small, the value of $\cos \alpha$ is nearly unity and $\sin \alpha$ is nearly zero, so that $M \cos \alpha$ equals M approximately, and $M \sin \alpha$ equals zero.

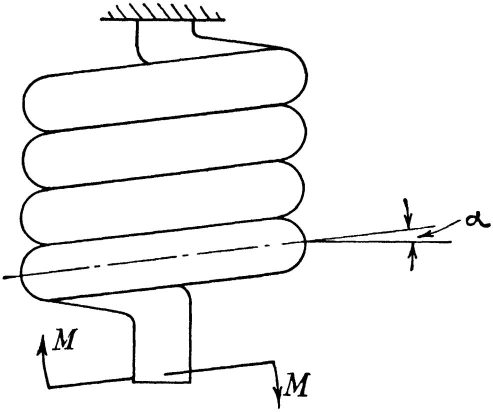


Fig. 7.9

Let ϕ be the angle of twist of the free end, then the work done equals $\frac{1}{2}M\phi$. The stress-strain energy equals $\frac{1}{2}M \,.\, ML/(EI)$, since the bending moment is constant.

Equating, $$\frac{1}{2}M\phi = \frac{1}{2}\frac{M^2L}{EI},$$

$$\therefore \phi = \frac{ML}{EI}.$$

The length $L = 2\pi Rn$, where n = number of coils,

and $I = \frac{1}{64}\pi d^4$, where d = diameter of wire.

Hence $$\phi = \frac{M \,.\, 2\pi Rn}{E\frac{1}{64}\pi d^4}$$

$$= \frac{128MRn}{Ed^4}.$$

The stress-strain energy $= \frac{\frac{1}{2}M^2L}{EI}$,

but $\frac{M}{I} = \frac{f}{m}$,

where $f =$ bending stress,

$m =$ distance of material from neutral surface $= r$.

$$\text{Hence stress-strain energy} = \tfrac{1}{2}\frac{f^2I^2}{r^2}\cdot\frac{L}{EI}$$

$$= \tfrac{1}{2}\frac{f^2IL}{r^2E}$$

$$= \tfrac{1}{2}\frac{f^2\frac{1}{4}\pi r^4L}{r^2E}$$

$$= \tfrac{1}{8}\frac{f^2}{E}\times \pi r^2L$$

$$= \tfrac{1}{8}\frac{f^2}{E}\times \text{volume of wire.}$$

7. Open-coiled Spring with Axial Torque.

In this case both the terms $M\cos\alpha$ and $M\sin\alpha$ must be used.

$$\text{Work done} = \tfrac{1}{2}M\phi.$$

$$\text{Stress-strain energy} = \tfrac{1}{2}M\cos\alpha\,.\,\frac{LM\cos\alpha}{EI} + \tfrac{1}{2}M\sin\alpha\,.\,\frac{LM\sin\alpha}{CI_0}$$

$$= \tfrac{1}{2}\frac{M^2L\cos^2\alpha}{EI} + \tfrac{1}{2}\frac{M^2L\sin^2\alpha}{CI_0}$$

$$= \tfrac{1}{2}M^2L\left[\frac{\cos^2\alpha}{EI} + \frac{\sin^2\alpha}{CI_0}\right].$$

But $L = 2\pi Rn\sec\alpha$.

$$\text{Hence stress-strain energy} = \tfrac{1}{2}M^2 2\pi Rn\sec\alpha\left[\frac{\cos^2\alpha}{EI} + \frac{\sin^2\alpha}{CI_0}\right]$$

$$= M^2\pi Rn\sec\alpha\left[\frac{\cos^2\alpha}{EI} + \frac{\sin^2\alpha}{CI_0}\right].$$

Equating work done to stress-strain energy,

$$\tfrac{1}{2}M\phi = M^2\pi Rn \sec\alpha\left[\frac{\cos^2\alpha}{EI} + \frac{\sin^2\alpha}{CI_0}\right]$$

$$= \frac{M^2\pi Rn \sec\alpha}{\frac{1}{32}\pi d^4}\left[\frac{2\cos^2\alpha}{E} + \frac{\sin^2\alpha}{C}\right],$$

since $I_0 = 2I = \frac{1}{32}\pi d^4$.

Hence $$\phi = \frac{64MRn \sec\alpha}{d^4}\left[\frac{2\cos^2\alpha}{E} + \frac{\sin^2\alpha}{C}\right].$$

Example 3.—A close-coiled helical spring of circular section extends 1 in. when subjected to an axial load W, and there is an angular rotation of one radian when a torque T is independently applied about the axis of the spring. If Poisson's ratio is $1/m$, and the mean diameter of the coil is D, show that

$$\frac{T}{W} = \frac{D^2(m+1)}{4m}.$$

Determine Poisson's ratio for the material of a spring of 3 in. mean diameter of coil, if 55 lb. extends the spring 5·1 in. and a torque of 30 lb.-in. produces an angular rotation of 60°. (London Univ.)

When a close-coiled spring is axially loaded, the deflection $\delta = \frac{64WR^3n}{\quad}$

where W = axial load, R = coil radius, n = number of coi of wire, C = modulus of rigidity.

With an axial torque, the angular rotation of

$$\phi = \frac{128TRn}{Ed^4},$$

where T = torque, E = modulus of elasticity.

Substituting for δ and ϕ,

$$1 = \frac{64WR^3n}{Cd^4},$$

$$1 = \frac{128TRn}{Ed^4},$$

$$\therefore \frac{T}{W} = \frac{64R^2E}{128C} = \frac{1}{2}\frac{R^2E}{C}.$$

But $\frac{1}{C} = \frac{2}{E}\left(1 + \frac{1}{m}\right)$ from the relationship between elastic constants (see *Materials and Structures*, p. 137).

Hence
$$\frac{T}{W} = \frac{R^2}{2} E \times \frac{2}{E}\left(1 + \frac{1}{m}\right)$$
$$= R^2\left(1 + \frac{1}{m}\right)$$
$$= \frac{D^2}{4m}(m + 1), \text{ where } D = 2R.$$

When the extension is 5·4 in., the axial load is 55 lb., therefore for 1 in. extension the load is 10·18 lb. An angular rotation of $60\pi/180$ radians is produced by a twisting moment of 30 lb.-in., therefore a rotation of 1 radian is produced by a torque of 28·66 lb.-in.

Substituting in equation $\frac{T}{W} = \frac{D^2}{4m}(m + 1)$,

$$\frac{28 \cdot 66}{10 \cdot 18} = \frac{3^2}{4m}(m + 1).$$
$$\therefore\ 1 \cdot 25 = \frac{m + 1}{m}.$$
$$\therefore\ 1 \cdot 25m = m + 1,$$

or
$$0 \cdot 25m = 1,$$
$$\therefore\ m = 4.$$

Hence
$$\text{Poisson's ratio} = \frac{1}{m} = \tfrac{1}{4}.$$

8. Combined Bending and Torsion.

A typical example, in which a complex system of stressing applies, is a shaft subjected to a twisting moment and to a bending moment, on account of its being supported on bearings and loaded transversely by pulleys. The normal theories of failure may be applied to this problem, but the theories generally used are (i) maximum stress for shafts of brittle material, (ii) *A*. maximum shear stress, or *B*. maximum strain energy for shafts of ductile material.

If the shaft is subjected to a twisting moment T, then the shear stress

$$f_s = \frac{Tr}{I_0},$$

where I_0 = polar second moment of area $= \frac{1}{32}\pi d^4$, and $r = \frac{1}{2}d$.

When the bending moment on the shaft is M, the bending stress

$$f_t = \frac{Mr}{I},$$

where I = second moment of area about the diameter $= \frac{1}{64}\pi d^4$.

The stress system on a shaft subjected to bending and torsion is shown in fig. 7.10.

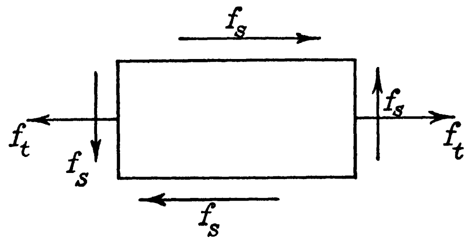


Fig. 7.10

(a) *Maximum-stress theory.*

The principal stresses are given by equations

$$p_{n_1} = \tfrac{1}{2}\{(p_x + p_y) + \sqrt{(p_x - p_y)^2 + 4f_s^2}\},$$

$$p_{n_2} = \tfrac{1}{2}\{(p_x + p_y) - \sqrt{(p_x - p_y)^2 + 4f_s^2}\},$$

when tensile or compressive stresses on two planes at right angles and a shear stress f_s act (see p. 173).

In the shaft problem there is only one tensile or compressive stress, so that $p_y = 0$, $p_x = f_t = Mr/I$, and $f_s = Tr/I_0$.

$$\therefore\ p_n = \tfrac{1}{2}\left\{\frac{Mr}{I} \pm \sqrt{\left[\left(\frac{Mr}{I}\right)^2 + 4\left(\frac{Tr}{I_0}\right)^2\right]}\right\}$$

$$= \tfrac{1}{2}\frac{r}{I}\{M \pm \sqrt{(M^2 + T^2)}\},\ \text{since } I = \tfrac{1}{2}I_0.$$

$$\therefore\ \frac{p_n I}{r} = \tfrac{1}{2}\{M \pm \sqrt{(M^2 + T^2)}\}.$$

$$\therefore\ M_e = \tfrac{1}{2}\{M \pm \sqrt{(M^2 + T^2)}\}.$$

$$\therefore\ M_e = \tfrac{1}{2}M + \tfrac{1}{2}\sqrt{(M^2 + T^2)},$$

where M_e is the equivalent bending moment $= p_n I/r$, and the numerical maximum only is considered.

(b) *Maximum-strain theory.*

When direct strains take place on two planes at right angles to one another and a shear strain also takes place, the maximum strains

or principal strains are given by

$$e_1 = \tfrac{1}{2}\{(e_x + e_y) + \sqrt{(e_x - e_y)^2 + \phi^2}\}$$

and
$$e_2 = \tfrac{1}{2}\{(e_x + e_y) - \sqrt{(e_x - e_y)^2 + \phi^2}\},$$

where e_x and e_y are the direct strains and ϕ is the shear strain (see p. 180).

In the shaft problem, $e_x = f_t/E$, $e_y = -f_t/(mE)$, and $\phi = f_s/C$, where E = modulus of elasticity, and C = modulus of rigidity.

$$\therefore\ e = \tfrac{1}{2}\left\{\left(\frac{f_t}{E} - \frac{f_t}{mE}\right) \pm \sqrt{\left[\left(\frac{f_t}{E} + \frac{f_t}{mE}\right)^2 + \frac{f_s^2}{C^2}\right]}\right\}$$

$$= \frac{1}{2E}\left\{f_t\left(1 - \frac{1}{m}\right) \pm \sqrt{\left[f_t^2\left(1 + \frac{1}{m}\right)^2 + 4f_s^2\left(1 + \frac{1}{m}\right)^2\right]}\right\},$$

$$\text{since } C = \frac{E}{2\left(1 + \frac{1}{m}\right)},$$

$$= \frac{1}{2E}\left\{\frac{Mr}{I}\left(1 - \frac{1}{m}\right) \pm \left(1 + \frac{1}{m}\right)\sqrt{\left[\left(\frac{Mr}{I}\right)^2 + 4\left(\frac{Tr}{I_0}\right)^2\right]}\right\}$$

$$= \frac{r}{2EI}\left\{M\left(1 - \frac{1}{m}\right) \pm \left(1 + \frac{1}{m}\right)\sqrt{[M^2 + T^2]}\right\},$$

$$\text{since } I = \tfrac{1}{2}I_0.$$

$$\therefore\ eE = \frac{r}{2I}\left\{M\left(1 - \frac{1}{m}\right) \pm \left(1 + \frac{1}{m}\right)\sqrt{[M^2 + T^2]}\right\}.$$

But $eE = \text{stress} = f.$

$$\therefore\ f\frac{2I}{r} = M\left(1 - \frac{1}{m}\right) \pm \left(1 + \frac{1}{m}\right)\sqrt{[M^2 + T^2]},$$

$$\therefore\ M_e = \tfrac{1}{2}M\left(1 - \frac{1}{m}\right) + \tfrac{1}{2}\left(1 + \frac{1}{m}\right)\sqrt{[M^2 + T^2]},$$

where M_e is the equivalent bending moment, and the positive or numerical maximum only is considered.

If $1/m = \tfrac{1}{4}$, then

$$M_e = \tfrac{1}{2}M(1 - \tfrac{1}{4}) + \tfrac{1}{2}(1 + \tfrac{1}{4})\sqrt{(M^2 + T^2)}$$

$$= \tfrac{3}{8}M + \tfrac{5}{8}\sqrt{(M^2 + T^2)}.$$

(*c*) *Maximum-shear-stress or stress-difference theory.*

The principal stresses are

$$p_{n_1} = \tfrac{1}{2}\{(p_x + p_y) + \sqrt{(p_x - p_y)^2 + 4f_s^2}\}$$

and

$$p_{n_2} = \tfrac{1}{2}\{(p_x + p_y) - \sqrt{(p_x - p_y)^2 + 4f_s^2}\}.$$

Therefore the stress difference

$$p_{n_1} - p_{n_2} = \sqrt{[(p_x - p_y)^2 + 4f_s^2]},$$

where $p_y = 0$.

$$\therefore\ p = \sqrt{\left[\left(\frac{Mr}{I}\right)^2 + 4\left(\frac{Tr}{I_0}\right)^2\right]}$$

$$= \frac{r}{I}\sqrt{[M^2 + T^2]}.$$

$$\therefore\ M_e = \sqrt{[M^2 + T^2]},$$

where M_e is the equivalent bending moment $= pI/r$. If the above were deduced from the maximum shear stress it would become

$$q = \tfrac{1}{2}(p_{n_1} - p_{n_2})$$

$$= \tfrac{1}{2}\sqrt{\left[\left(\frac{Mr}{I}\right)^2 + 4\left(\frac{Tr}{I_0}\right)^2\right]}$$

$$= \tfrac{1}{2}\,\frac{r}{I}\sqrt{[M^2 + T^2]}$$

$$= \frac{r}{I_0}\sqrt{[M^2 + T^2]},$$

or

$$T_e = \sqrt{[M^2 + T^2]},$$

where T_e is the equivalent twisting moment $= qI_0/r$. If it is recognized that $q = \tfrac{1}{2}p$, the two equations are equivalent.

(*d*) *Maximum-strain-energy theory.*

For a two-dimensional stress condition,

the strain energy per unit volume $= U$

$$= \frac{1}{2E}\left\{p_1^2 + p_2^2 - \frac{2}{m}(p_1 p_2)\right\}$$

(see p. 187).

The maximum strain energy will be given when p_1 and p_2 are the maximum or principal stresses, i.e. $p_1 = p_{n_1}$ and $p_2 = p_{n_2}$.

Now
$$p_{n_1} = \tfrac{1}{2}\{(p_x + p_y) + \sqrt{(p_x - p_y)^2 + 4f_s^2}\},$$
$$p_{n_2} = \tfrac{1}{2}\{(p_x + p_y) - \sqrt{(p_x - p_y)^2 + 4f_s^2}\},$$

where $p_y = 0$ in the case under consideration. Hence maximum strain energy
$$U = \frac{1}{2E}\left\{p_{n_1}^2 + p_{n_2}^2 - \frac{2}{m}p_{n_1}p_{n_2}\right\},$$

where
$$p_{n_1} = \tfrac{1}{2}\{p_x + \sqrt{[p_x^2 + 4f_s^2]}\}$$

and
$$p_{n_2} = \tfrac{1}{2}\{p_x - \sqrt{[p_x^2 + 4f_s^2]}\}.$$

$$\therefore\ U = \frac{1}{2E}\left\{p_x^2 + 2\frac{(m+1)}{m}f_s^2\right\}$$
$$= \frac{1}{2E}\left\{f_t^2 + 2\frac{(m+1)}{m}f_s^2\right\}, \text{ where } f_t = p_x,$$
$$= \frac{1}{2E}\left\{\left(\frac{Mr}{I}\right)^2 + 2\frac{(m+1)}{m}\left(\frac{Tr}{I_0}\right)^2\right\}$$
$$= \frac{r^2}{2EI^2}\left\{M^2 + \frac{(m+1)}{2m}T^2\right\}.$$

If f be the stress at the limit of proportionality, then
$$U = \tfrac{1}{2}f\frac{f}{E}$$
$$= \tfrac{1}{2}\frac{f^2}{E}.$$
$$\therefore\ \frac{f^2}{2E} = \frac{r^2}{2EI^2}\left\{M^2 + \frac{(m+1)}{2m}T^2\right\}.$$
$$\therefore\ f = \frac{r}{I}\sqrt{\left\{M^2 + \frac{(m+1)}{2m}T^2\right\}},$$

or
$$M_e = \sqrt{\left\{M^2 + \frac{(m+1)}{2m}T^2\right\}},$$

where M_e is the equivalent bending moment $= fI/r$.

If $1/m = \tfrac{1}{4}$, $\qquad M_e = \sqrt{\{M^2 + \tfrac{5}{8}T^2\}}$.

Summary.

(*a*) $M_e = \frac{1}{2}M + \frac{1}{2}\sqrt{M^2 + T^2}$.

(*b*) $M_e = \frac{3}{8}M + \frac{5}{8}\sqrt{M^2 + T^2}$, if $1/m = \frac{1}{4}$.

(*c*) $M_e = \sqrt{M^2 + T^2}$.

(*d*) $M_e = \sqrt{M^2 + \frac{5}{8}T^2}$, if $1/m = \frac{1}{4}$.

Example 4.—A twisting moment of 50 tons-ft. and a bending moment of 30 tons-ft. act simultaneously on a shaft. If the allowable tensile stress in the material is 6 tons per sq. in., find the diameter of the shaft using the four theories of failure ($1/m = \frac{1}{4}$).

(i) Maximum-stress theory.

$$M_e = \tfrac{1}{2}M + \tfrac{1}{2}\sqrt{(M^2 + T^2)},$$

where $M = 30$ tons-ft., and $T = 50$ tons-ft.,

$$= \tfrac{1}{2} \times 30 + \tfrac{1}{2}\sqrt{(30^2 + 50^2)}$$
$$= 15 + 29{\cdot}2$$
$$= 44{\cdot}2 \text{ tons-ft.}$$

$$\therefore \frac{fI}{r} = 44{\cdot}2 \times 12 \text{ tons-in.},$$

where $f = 6$ tons per sq. in., $I = \frac{1}{64}\pi d^4$, $r = \frac{1}{2}d$.

$$\therefore 6 \times \tfrac{1}{32}\pi d^3 = 44{\cdot}2 \times 12,$$
$$\therefore d^3 = 900,$$
$$\therefore d = 9{\cdot}65 \text{ in.}$$

(ii) Maximum-strain theory.

$$M_e = \tfrac{1}{2}M\left(1 - \frac{1}{m}\right) + \tfrac{1}{2}\left(1 + \frac{1}{m}\right)\sqrt{(M^2 + T^2)}$$
$$= \tfrac{1}{2} \times 30(1 - \tfrac{1}{4}) + \tfrac{1}{2}(1 + \tfrac{1}{4})\sqrt{(30^2 + 50^2)}$$
$$= 11{\cdot}25 + \tfrac{5}{8} \times 58{\cdot}4$$
$$= 11{\cdot}25 + 36{\cdot}5$$
$$= 47{\cdot}75 \text{ tons-ft.}$$
$$= 573 \text{ tons-in.}$$

$$\therefore \frac{fI}{r} = 573.$$

$$\therefore 6 \times \tfrac{1}{32}\pi d^3 = 573.$$
$$\therefore d^3 = 975.$$
$$\therefore d = 9{\cdot}9 \text{ in.}$$

(iii) Maximum-stress-difference theory.

$$\begin{aligned} M_e &= \sqrt{(M^2 + T^2)} \\ &= \sqrt{(30^2 + 50^2)} \\ &= 58{\cdot}4 \text{ tons-ft.} \\ &\equiv 700{\cdot}8 \text{ tons-in.} \end{aligned}$$

$$\therefore \frac{fI}{r} = 700{\cdot}8.$$

$$\therefore 6 \times \tfrac{1}{32}\pi d^3 = 700{\cdot}8.$$

$$\therefore d^3 = 1200.$$

$$\therefore d = 10{\cdot}6 \text{ in.}$$

(iv) Maximum-strain-energy theory.

$$\begin{aligned} M_e &= \sqrt{\left[M^2 + \frac{(m+1)}{2m} T^2\right]} \\ &= \sqrt{(30^2 + \tfrac{5}{8} \times 50^2} \\ &= 49{\cdot}6 \text{ tons-ft.} \\ &\equiv 595 \text{ tons-in.} \end{aligned}$$

$$\therefore \frac{fI}{r} = 595.$$

$$\therefore 6 \times \tfrac{1}{32}\pi d^3 = 595.$$

$$\therefore d^3 = 1009.$$

$$\therefore d = 10{\cdot}03 \text{ in.}$$

The maximum-stress-difference or maximum-shear-stress theory requires the largest shaft, consequently it is the safest theory to use. There are insufficient experimental data to justify the use of the maximum-strain-energy theory, but in the case of brittle materials there is sufficient experimental evidence to show that they fail within the range of the maximum-stress theory.

When the bending is caused by two or more loads which are not acting in the same direction, the bending moments for each separate load must be combined vectorially in order to obtain the maximum value.

✓ *Example* 5.—A shaft is supported over a span of 12 ft. and is used to transmit 20 h.p. at 100 r.p.m. The power is transmitted from the shafts through two pulleys A and B. The pulleys are placed 6 ft. and 9 ft. respectively from the left-hand support. The total belt pull on pulley A is 1500 lb. in a horizontal direction, whilst the total pull on B is 1000 lb. in a direction making 70° with the pull on A. Using the maximum-shear-

stress theory, determine the diameter of the shaft if the allowable tensile stress is 6 tons per sq. in.

$$\text{Twisting moment} = T = \frac{(\text{horse-power}) \times 33{,}000}{2\pi \times \text{r.p.m.}}$$

$$= \frac{20 \times 33{,}000}{2\pi \times 100}$$

$$= 1052 \text{ lb.-ft.}$$

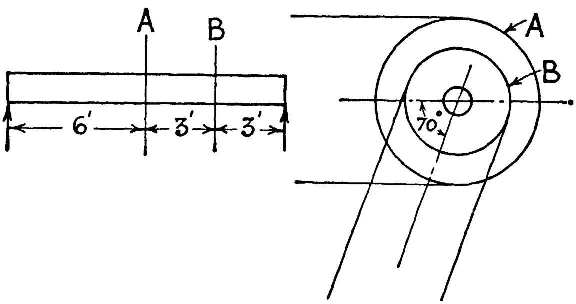


Fig. 7.11

For pulley A the maximum bending moment caused by the pull

$$= \tfrac{1}{4} \times 1500 \times 12$$

$$= 4500 \text{ lb.-ft.}$$

The bending-moment diagram for the 1500 lb. load is given in fig. 7.12.

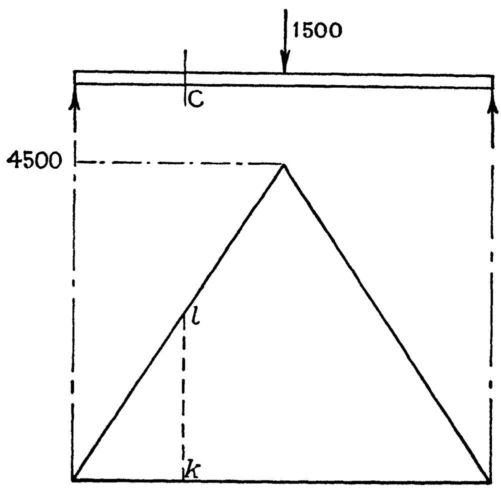


Fig. 7.12

For pulley B the maximum bending moment caused by the pull

$$= \tfrac{1}{12} \times 1000 \times 3 \times 9$$
$$= 2250 \text{ lb.-ft.}$$

The bending-moment diagram for the 1000 lb. load is given in fig. 7.13. The bending moments act in the directions shown in fig. 7.14. The net bending moment at any section is obtained by combining the separate bending moments vectorially. The net bending moment for section C is bending moment *kl* added vectorially to *mn* as shown in fig. 7.15. Hence *ko* is the net bending moment at C. A combined bending-moment diagram can be built up as is shown in fig. 7.16.

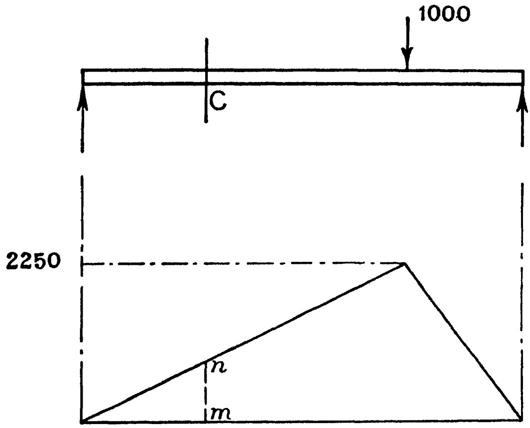


Fig. 7.13

The maximum bending occurs at A and is 5250 lb.-ft. Substituting in the formula,

$$M_e \quad \sqrt{(M^2 + T^2)}$$
$$= \sqrt{(5250^2 + 1052^2)}$$
$$= 5350 \text{ lb.-ft.}$$
$$= 64{,}200 \text{ lb.-in.}$$

But

$$M_e = \frac{fI}{r}$$
$$= \frac{6 \times 2240 \times \frac{1}{64}\pi d^4}{\frac{1}{2}d}$$
$$= \frac{6 \times 2240 \times \pi d^3}{32}.$$
$$\therefore\ d^3 = \frac{64{,}200 \times 32}{6 \times 2240 \times \pi}$$
$$= 48{\cdot}6.$$
$$\therefore\ d = 3{\cdot}65 \text{ in.}$$

If the bending moments produced by the separate loads are of opposite sign, then the directions in which the one set acts must be reversed before adding vectorially.

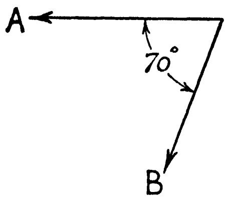


Fig. 7.14

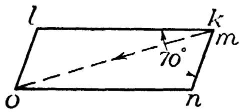


Fig. 7.15

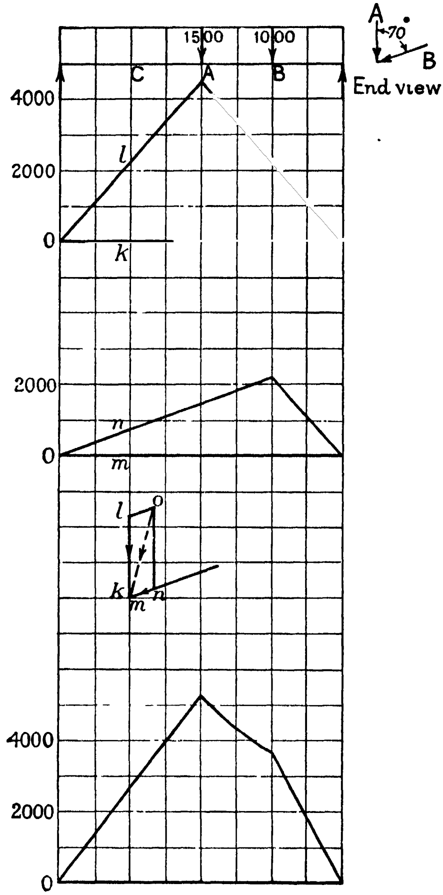


Fig. 7.16

9. End Thrust additional to Bending and Torsion.

A problem involving end thrust will be solved by one of the theories of failure, preferably the maximum-stress-difference theory. The bending stresses and the direct-thrust stress will be added algebraically in order to give the value of f_t which is used in the various theories. The bending stress will be Mr/I, whilst the direct stress will be P/A.

$$\text{Net stress } f_t = \frac{Mr}{I} + \frac{P}{A} \text{ compressive,}$$

$$\text{or } \frac{Mr}{I} - \frac{P}{A} \text{ tensile, if positive.}$$

Example 6.—The twisting moment on a propeller shaft is 200 tons-ft., whilst the maximum bending moment is 45 tons-ft. If there is a direct thrust of 25 tons, and the shaft is hollow with an external diameter of 15 in. and an internal diameter of $7\frac{1}{2}$ in., find the maximum tensile stress in the material, using (*a*) maximum-stress, (*b*) maximum-strain, (*c*) maximum-stress-difference, (*d*) maximum-strain-energy theories.

Take $\frac{1}{m} = \frac{1}{4}$.

$$\text{Shear stress } f_s = \frac{Tr}{I_0},$$

where

$$T = 200 \text{ tons-ft.} \equiv 2400 \text{ tons-in.},$$

$$I_0 = \frac{1}{32}\pi(D^4 - d^4)$$

$$= \frac{1}{32}\pi(15^4 - 7\tfrac{1}{2}^4)$$

$$= 4660 \text{ in.}^4 \text{ units},$$

$$r = 7{\cdot}5 \text{ in.}$$

$$\therefore f_s = \frac{2400 \times 7{\cdot}5}{4660}$$

$$= 3{\cdot}86 \text{ tons per sq. in.}$$

$$\text{Bending stress} = \frac{Mr}{I},$$

where

$$M = 45 \text{ tons-ft.} = 540 \text{ tons-in.}$$

$$I = \frac{1}{64}\pi(D^4 - d^4)$$

$$= \frac{1}{64}\pi(15^4 - 7{\cdot}5^4)$$

$$= 2330 \text{ in.}^4 \text{ units},$$

$$r = 7{\cdot}5 \text{ in.}$$

$$\therefore \text{Bending stress} = \frac{540 \times 7{\cdot}5}{2330}$$

$$= 1{\cdot}74 \text{ tons per sq. in.}$$

Direct compressive stress $= \dfrac{P}{A}$,

where $P = 25$ tons,

$$A = \tfrac{1}{4}\pi(D^2 - d^2)$$
$$= \tfrac{1}{4}\pi(15^2 - 7\cdot5^2)$$
$$= 133 \text{ sq. in.}$$

$\therefore$ Compressive stress $= \cdot19$ ton per sq. in.

Net direct stress $f_t = 1\cdot74 + \cdot19$

$= 1\cdot93$ tons per sq. in. compressive.

(*a*) Maximum-principal-stress theory

$$= \tfrac{1}{2}\{f_t + \sqrt{(f_t^2 + 4f_s^2)}\}$$
$$= \tfrac{1}{2}\{1\cdot93 + \sqrt{(1\cdot93^2 + 4 \times 3\cdot86^2)}\},$$
$$= \tfrac{1}{2}\{(1\cdot93 + \sqrt{(3\cdot72 + 59\cdot6)}\}$$
$$= \tfrac{1}{2}\{1\cdot93 + 7\cdot95\}$$
$$= 4\cdot94 \text{ tons per sq. in.}$$

(*b*) Maximum-strain theory:

$$\text{stress} = \text{strain} \times E = \tfrac{1}{2}\left\{f_t\left(1 - \frac{1}{m}\right) + \sqrt{\left(f_t^2\left(1 + \frac{1}{m}\right)^2 + 4f_s^2\left(1 + \frac{1}{m}\right)^2\right]}\right\}$$
$$= \tfrac{1}{2}\{1\cdot93(1 - \tfrac{1}{4}) + (1 + \tfrac{1}{4})\sqrt{[1\cdot93^2 + 4 \times 3\cdot86^2]}\}$$
$$= \tfrac{1}{2}\{1\cdot45 + 1\cdot25 \times 7\cdot95\}$$
$$= 5\cdot69 \text{ tons per sq. in.}$$

(*c*) Maximum-stress-difference theory:

$$\text{stress} = \sqrt{(f_t^2 + 4f_s^2)}$$
$$= \sqrt{(1\cdot93^2 + 4 \times 3\cdot86^2)}$$
$$= 7\cdot95 \text{ tons per sq. in.}$$

(*d*) Maximum-strain-energy theory:

$$U = \frac{1}{2E}\left\{f_t^2 + \frac{2(m+1)}{m}f_s^2\right\} \text{ per unit volume.}$$

$$\therefore \tfrac{1}{2}\frac{f^2}{E} = \frac{1}{2E}\left\{f_t^2 + \frac{2(m+1)}{m}f_s^2\right\}.$$

$$\therefore f = \sqrt{\left[f_t^2 + \frac{2(m+1)}{m} f_s^2\right]}$$

$$= \sqrt{\left[1{\cdot}93^2 + \frac{2(4+1)}{4} \times 3{\cdot}86^2\right]}$$

$$= \sqrt{[3{\cdot}72 + 2{\cdot}5 \times 14{\cdot}90]}$$

$$= \sqrt{40{\cdot}97}$$

$$= 6{\cdot}40 \text{ tons per sq. in.}$$

EXAMPLES

1. A solid steel shaft, 6 in. diameter, is to be replaced by a hollow shaft having three-quarters of the weight and made of steel in which the working stress is 25 per cent greater than that employed for the solid shaft. Assuming the steels have equal density and shear modulus, calculate the internal and external diameters of the hollow shaft, and compare its rigidity (i.e. torque per unit angle of twist) with that for the solid shaft. (London Univ.) [2·1 in., 5·6 in.; 74·5 %.]

✓ 2. An alloy solid shaft, $2\frac{1}{2}$ in. diameter, is to be coupled in series with a hollow steel shaft of the same external diameter. Find the internal diameter of the steel shaft if the angle of twist per unit length is to be 80 per cent of that of the alloy shaft for the same torque.

Determine the horse-power that can be safely transmitted at a speed of 800 r.p.m. if the limits of shearing stress are 3 and 5 tons per sq. in. in the alloy and steel respectively. $C_{\text{steel}} = 2\frac{1}{2} \times C_{\text{alloy}}$. (London Univ.) [2·1 in; 218.]

✓ 3. A solid alloy shaft of 2 in. diameter is to be coupled in series with a hollow steel shaft of the same external diameter. Find the internal diameter of the steel shaft if the angle of twist per unit length is to be 75 per cent of that of the alloy shaft.

Determine the speed at which the shafts are to be driven to transmit 250 h.p. if the limits of shearing stress are to be 3·5 and 5 tons per sq. in. in the alloy and steel respectively. Modulus of rigidity for steel = 2·2 × modulus of rigidity for alloy. (London Univ.) [1·585 in., 1480.]

4. For a square-section shaft in torsion it can be proved that

$$\theta = \frac{7{\cdot}11T}{Cs^4}, \text{ and } f_s = \frac{T}{{\cdot}208s^3},$$

where θ is the angle of twist per unit length, T is the twisting moment, C the shear modulus, s the length of one side of the square, and f_s the maximum shear stress in the material. Derive formulæ for the maximum stress in the material and the deflection of a closely coiled helical spring formed of square-section material.

Such a spring is required to carry a safe load of 3000 lb. and to deflect $1\frac{1}{4}$ in. per 1000 lb. load. The mean diameter of the coils is to be 5 times the length of the square side. Calculate suitable dimensions for the spring if the safe stress is 60,000 lb. per sq. in. and $C = 11{\cdot}5 \times 10^6$ lb. per sq. in. Work to the nearest $\frac{1}{16}$ in. for the value of s. (London Univ.)

5. A close-coiled helical spring is to have a stiffness of 5 lb. per in. of compression, a maximum load of 9 lb., and a maximum shearing stress of 18,000 lb. per sq. in. The solid length of the spring (i.e. when the coils are touching) is to be 1·8 in. Find the diameter of the wire, the mean radius of the coils and the number of coils required. Modulus of rigidity $= 6 \times 10^6$ lb. per sq. in. (London Univ.) [0·1145 in.; 0·59 in.; 15·7.]

6. A close-coiled helical spring has a mean diameter of 6 in. and is wound from steel rod of diameter 1 in. The stiffness of the spring is such that a force of $\frac{1}{3}$ ton extends it 1 in. Find the necessary number of turns and the maximum load which can be suspended from the spring without the shear stress in the rod exceeding 20 tons per sq. in. $C = 12 \times 10^6$ lb. per sq. in. [9·35; 1·31 tons.]

7. An open-coiled helical spring of round wire of diameter d has a mean coil diameter D with n complete coils. The coils make an angle α with planes perpendicular to the axis of the spring. If the modulus of elasticity is E, and of rigidity C, derive an expression for the extension due to an axial load w in terms of the symbols given. From this expression derive the value for the extension of a close-coiled spring.

If the close-coiled-spring formula is used in finding the extension of an open-coiled spring, determine the maximum value of α if the error in the value of the extension is not to exceed one per cent. $E = 2{\cdot}5\, C$. (London Univ.) [8° 6′.]

8. A circular shaft of external radius r and diametral moment of inertia I is subjected to a bending moment $M \cos \alpha$ and a twisting moment $M \sin \alpha$. Show that the magnitude of the maximum shearing stress has a constant value $\frac{1}{2}Mk^2$ for all values of α, where $k^2 = r/I$.

Show clearly on a diagram the planes on which the maximum shearing stresses occur when $\alpha = 45°$. (London Univ.)

9. In a circular shaft subjected to an axial twisting moment T and a bending moment M, show that, when $M = 1{\cdot}2T$, the ratio of the maximum shearing stress to the greater principal stress is approximately 0·566. (London Univ.)

10. A horizontal shaft is securely fixed at each end and has a free length of 6 ft. Two brackets, with arms horizontal and perpendicular to the axis of the shaft, are fixed to the shaft at points 2 and 4 ft. from one of the fixed ends. Vertical loads of 1 and $1\frac{1}{2}$ tons, at distances of 8 and 6 in. respectively from the axis of the shaft, are suspended from the arms, and both tend to twist the shaft in the same direction. Determine the end fixing moments to resist bending and twisting, and find the maximum shearing stress in the shaft if the diameter is 3 in. (London Univ.)

11. A mild-steel shaft is subjected to a torque of 10 tons-in. and to a bending moment of 6 tons-in. Calculate the equivalent torque on the

shaft and necessary diameter for a maximum shearing stress of 5 tons per sq. in. (i) according to the maximum-shearing-stress theory, (ii) according to the strain-energy theory. Poisson's ratio = 0·304. (London Univ.)

[(i) 11·65 tons.-in.; 2.28 in.; (ii) 12·48 tons.-in.; 2·33 in.]

12. A horizontal circular shaft of diameter d and diametral moment of inertia I is subjected to a bending moment $M \cos \theta$ in a vertical plane and to an axial twisting moment $M \sin \theta$. Show that the principal stresses at the ends of a vertical diameter are $\frac{1}{2}Mk(\cos \theta \pm 1)$, where $k = d/2I$.

If strain energy is the criterion of failure, show that

$$s = \frac{s_0\sqrt{2}}{\sqrt{\{\cos^2 \theta(1 - \sigma) + (1 + \sigma)\}}},$$

where s = maximum shear stress, s_0 = maximum shearing stress in the special case when $\theta = 0$, and σ = Poisson's ratio. (London Univ.)

13. A solid shaft, 5 in. diameter, transmits 800 h.p. at 300 r.p.m. It is also subjected to a bending moment of 36 tons-in. and to an end thrust. If the maximum principal stress is limited to 5 tons per sq. in., determine the permissible end thrust.

Determine the position of the plane in which the principal stress acts, and draw a diagram showing the position of the plane relative to the torque and the plane of the bending moment applied to the shaft. (London Univ.)

[3·73 tons.; 31° 26′.]

14. An engine shaft is subjected simultaneously to a twisting moment (T) and a bending moment (M); show that the equivalent bending moment, on the maximum-direct-strain theory, is given by the formula

$$M_e = \frac{1}{2}\left\{\frac{m-1}{m} M + \frac{m+1}{m} \sqrt{(M^2 + T^2)}\right\},$$

where $1/m$ = Poisson's ratio.

A solid circular shaft, 6 in. in diameter, is subjected to a bending moment of 60 tons-in. and a twisting moment of 20 tons-in.; determine the maximum direct stress in the material due to the maximum direct strain induced, taking $m = 4$.

[2·92 tons per sq. in.]

CHAPTER VIII

Compound Stresses

The relationship between the four elastic constants has been found in *Materials and Structures*, Chap. VII, and in consequence, if two of the constants are found experimentally, the other two can be found by the use of the formulæ. Where C = modulus of rigidity, E = modulus of elasticity, $1/m$ = Poisson's ratio, K = modulus of bulk,

$$\frac{1}{C} = \frac{2}{E}\left(1 + \frac{1}{m}\right), \quad \ldots \ldots \quad (1)$$

$$K = \frac{E}{3\left(1 - \frac{2}{m}\right)}. \quad \ldots \ldots \quad (2)$$

From (1), $1/m$ cannot be less than -1, as C cannot be negative; from (2), $2/m$ cannot be greater than 1, as K is positive; therefore $1/m$ lies between -1 and $+\frac{1}{2}$.

By substituting for E from (2) in (1),

$$\frac{1}{C} = \frac{2\left(1 + \frac{1}{m}\right)}{3K\left(1 - \frac{2}{m}\right)}.$$

$$\therefore\ 3K\left(1 - \frac{2}{m}\right) = 2C\left(1 + \frac{1}{m}\right).$$

$$\therefore\ 3K - \frac{6K}{m} = 2C + \frac{2C}{m}.$$

$$\therefore\ \frac{2C}{m} + \frac{6K}{m} = 3K - 2C.$$

$$\therefore\ \frac{1}{m}(2C + 6K) = 3K - 2C,$$

or

$$\frac{1}{m} = \frac{3K - 2C}{6K + 2C}. \quad \ldots \ldots \quad (3)$$

By substituting for $1/m$ from (3) in (1),

$$\frac{1}{C} = \frac{2}{E}\left(1 + \frac{3K - 2C}{6K + 2C}\right).$$

$$\therefore\ E = 2C\left[\frac{6K + 2C + 3K - 2C}{6K + 2C}\right]$$

$$= \frac{9KC}{3K + C} \quad \ldots\ldots\ldots\ldots \quad (4)$$

Example 1.—A hollow shaft of 3 in. external diameter and 2 in. internal diameter twists through an angle of 0·6 degree in a length of 4 ft. when subjected to an axial twisting moment of 4 tons-in. Estimate the deflection at the centre of the shaft due to its own weight when placed in a horizontal position on supports 4 ft. apart. Weight of shaft = 13·2 lb. per foot length. Poisson's ratio = 0·3.

Applying formula $\frac{T}{I_0} = \frac{C\theta}{L}$,

where T = 4 tons-in. = 8960 lb.-in., $I_0 = \frac{1}{32}\pi(3^4 - 2^4) = 6{\cdot}38$ in.^{4} units, $\theta = \frac{\pi}{180} \times 0{\cdot}6$ radians, L = 48 in., we have

$$\frac{8960}{6{\cdot}38} = \frac{C \times \frac{\pi \times 0{\cdot}6}{180}}{48}.$$

$$\therefore\ C = \frac{8960 \times 48 \times 180}{6{\cdot}38 \times \pi \times 0{\cdot}6}$$

$$= 6{,}450{,}000 \text{ lb. per sq. in.}$$

Applying formula $\frac{1}{C} = \frac{2}{E}\left(1 + \frac{1}{m}\right)$,

where C = 6,450,000 lb. per sq. in., and $1/m = 0{\cdot}3$, we have

$$\frac{1}{6{,}450{,}000} = \frac{2}{E}(1 + 0{\cdot}3).$$

$$\therefore\ E = 2{\cdot}6 \times 6{,}450{,}000$$

$$= 16{,}780{,}000 \text{ lb. per sq. in.}$$

Deflection at the centre of a simply supported beam with a uniformly distributed load is given by

$$\delta = \frac{5}{384}\frac{wL^4}{EI},$$

where w = load per in. length = 1·1 lb. per in. length, L = 48 in., $I = \frac{1}{64}\pi(3^4 - 2^4) = 3{\cdot}19$ in.^{4} units, E = 16,780,000 lb. per sq. in.

$$\therefore\ \delta = \frac{5}{384}\frac{1{\cdot}1 \times 48^4}{16{,}780{,}000 \times 3{\cdot}19}$$

$$= {\cdot}00142 \text{ in.}$$

1. Stress Components on an Oblique Plane due to a Direct Stress on a Given Plane.

This case has been considered in *Materials and Structures*, but for completeness it is recapitulated here.

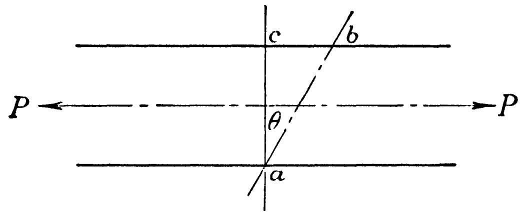


Fig. 8.1

Let A_1 = area of oblique plane ab,

A = area of normal plane ac,

θ = angle of obliquity from normal to main axis,

then $A_1 = A \sec\theta$.

$$\text{Direct stress } p = \frac{P}{A}.$$

$$\text{Normal stress to oblique plane } p_n = \frac{P_n}{A_1}$$

$$= \frac{P_n}{A\sec\theta}$$

$$= \frac{P_n \cos\theta}{A}.$$

$$\text{Shear stress on oblique plane } q = \frac{Q}{A_1}$$

$$= \frac{Q\cos\theta}{A}.$$

Resolving normal to ab,

$$P_n = P\cos\theta,$$
$$\therefore\ p_n A\sec\theta = pA\cos\theta,$$
$$\therefore\ p_n = p\cos^2\theta.$$

Resolving parallel to ab,

$$Q = P\sin\theta,$$
$$\therefore\ qA\sec\theta = pA\sin\theta,$$
$$\therefore\ q = p\sin\theta\cos\theta$$
$$= \tfrac{1}{2}p\sin 2\theta.$$

The resultant of p_n and q may be found. Let r be the resultant stress, then

$$r^2 = p_n^2 + q^2$$
$$= p^2\cos^4\theta + p^2\sin^2\theta\cos^2\theta$$
$$= p^2\cos^2\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= p^2\cos^2\theta.$$
$$\therefore\ r = p\cos\theta.$$

The maximum normal stress occurs when $\cos^2\theta$ is a maximum, i.e. when $\theta = 0$, then $p_n = p$.

The maximum shear stress occurs when $\sin 2\theta$ is a maximum, i.e. when $2\theta = 90°$ and $\theta = 45°$, then $q = \tfrac{1}{2}p$.

The maximum resultant occurs when $\theta = 0$, and therefore $r = p$.

Example 2.—A tie-bar has a rectangular section 4 in. by $1\tfrac{1}{2}$ in. and carries a load of 30 tons. Find the normal and tangential stresses on sections making 0°, 15°, 30°, 45°, 60°, 75° and 90° with the normal to the axis of the bar. Plot curves showing the relation of stresses and angles.

$$A = 4 \times 1\tfrac{1}{2} = 6 \text{ sq. in.}, \quad P = 30 \text{ tons},$$
$$\therefore\ p = \frac{P}{A} = 5 \text{ tons per sq. in.}$$

Applying formulæ $\quad p_n = p\cos^2\theta = 5\cos^2\theta$

and $\quad q = \tfrac{1}{2}p\sin 2\theta = 2\tfrac{1}{2}\sin 2\theta,$

we can obtain the following values:

θ	0°	15°	30°	45°	60°	75°	90°
p_n	5	4·66	3·75	2·5	1·25	0·335	0
q	0	1·25	2·16	2·5	2·16	1·25	0
r	5	4·83	4·33	3·54	2·5	1·294	0

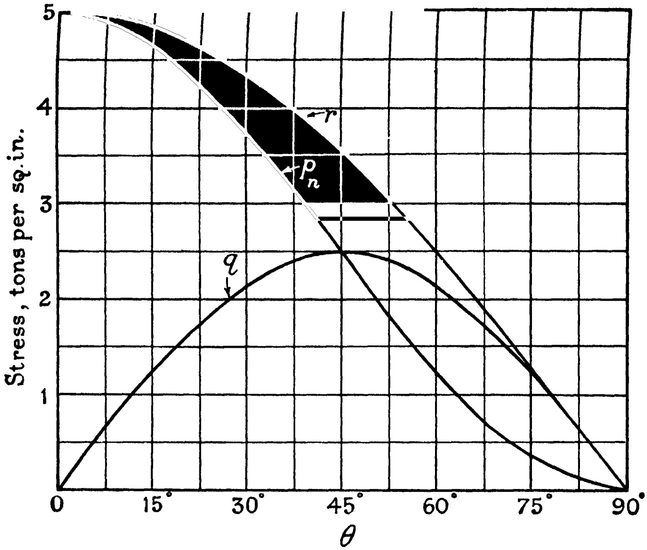


Fig. 8.2

2. Stress Components on an Oblique Plane due to a Shearing Stress on a Given Plane.

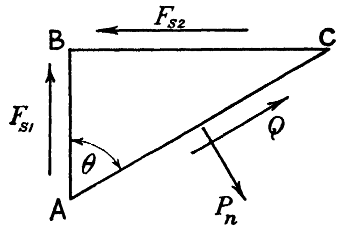


Fig. 8.3

Let A_1 = area of oblique plane AC,

A_2 = area of plane BC,

A = area of normal plane AB,

θ = angle of obliquity from normal to main axis,

then $A_1 = A \sec \theta$, and $A_2 = A \tan \theta$.

$$\text{Shear stress } f_s = \frac{F_{s_1}}{A} = \frac{F_{s_2}}{A_2}.$$

It has been shown in Chap. VII of *Materials and Structures* that shearing stresses on AB are accompanied by equal shearing stresses on BC.

Normal stress on oblique plane $p_n = \frac{P_n}{A_1}$

$$= \frac{P_n \cos\theta}{A}.$$

Shear stress on oblique plane $q = \frac{Q}{A_1}$

$$= \frac{Q\cos\theta}{A}.$$

Resolving F_{s_1} and F_{s_2} normal to AC,

$$P_n = F_{s_1}\sin\theta + F_{s_2}\cos\theta,$$

$$\therefore\ p_n A\sec\theta = f_s A\sin\theta + f_s A_2\cos\theta$$

$$= f_s A\sin\theta + f_s A\tan\theta\cos\theta.$$

$$\therefore\ p_n = f_s\sin\theta\cos\theta + f_s\sin\theta\cos\theta$$

$$= 2f_s\sin\theta\cos\theta$$

$$= f_s\sin 2\theta.$$

Resolving F_{s_1} and F_{s_2} parallel to AC,

$$Q = -F_{s_1}\cos\theta + F_{s_2}\sin\theta,$$

$$\therefore\ qA\sec\theta = -f_s A\cos\theta + f_s A\tan\theta\sin\theta.$$

$$\therefore\ q = -f_s\cos^2\theta + f_s\sin^2\theta$$

$$= f_s(\sin^2\theta - \cos^2\theta)$$

$$= -f_s\cos 2\theta.$$

p_n is a maximum when $\sin 2\theta$ is a maximum, i.e. when $\theta = 45°$ or $135°$
q is a maximum when $\theta = 0$ and $90°$.

Example 3.—A shaft, 2 in. diameter, is subjected to a pure twisting moment, so that the maximum shear stress is 5 tons per sq. in. Find the normal and tangential stresses on sections making 0, 15°, 30° 45°, 60°, 75° and 90° with the normal to the axis of the shaft. Plot curves showing the relation of stresses and angles. $f_s = 5$ tons per sq. in.

Applying formulæ $p_n = f_s\sin 2\theta$, $q = -f_s\cos 2\theta$, we have (fig. 8.4)

θ	0°	15°	30°	45°	60°	75°	90°
p_n	0	2·5	4·33	5	4·33	2·5	0
q	−5	−4·33	−2·5	0	2·5	4·33	5

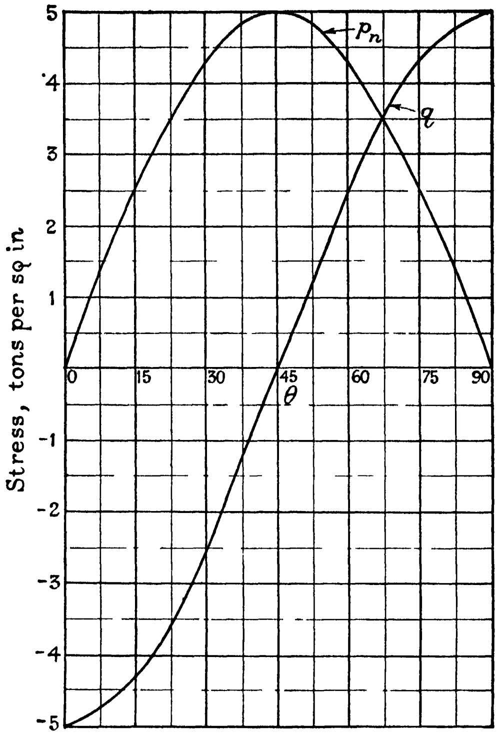


Fig 8.4

3. Two Perpendicular Normal Stresses.

(1) *Like stresses.*—Let total stresses P_y and P_x act on planes AB and BC respectively. If area of BC is A, then area of AB is $A \tan\theta$, and area of AC is $A \sec\theta$.

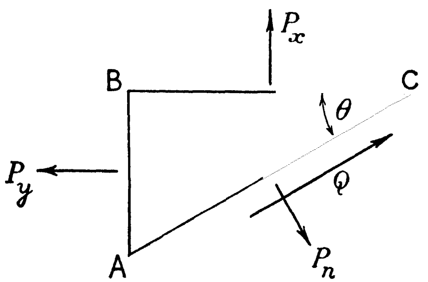


Fig. 8.5

$$p_x = \frac{P_x}{A},$$

$$p_y = \frac{P_y}{A \tan\theta}.$$

Normal stress on oblique plane $p_n = \dfrac{P_n}{A \sec\theta}$.

Shear stress on oblique plane $q = \dfrac{Q}{A \sec\theta}$.

Resolving P_y and P_x normal to AC,

$$P_n = P_x \cos\theta + P_y \sin\theta.$$

$$\therefore\ p_n A \sec\theta = p_x A \cos\theta + p_y A \tan\theta \sin\theta,$$

$$\therefore\ p_n = p_x \cos^2\theta + p_y \sin^2\theta.$$

Resolving P_y and P_x parallel to AC,

$$Q = P_x \sin\theta - P_y \cos\theta.$$

$$\therefore\ qA \sec\theta = p_x A \sin\theta - p_y A \tan\theta \cos\theta,$$

$$\therefore\ q = p_x \sin\theta\cos\theta - p_y \sin\theta \cos\theta$$
$$= \tfrac{1}{2}(p_x - p_y)\sin 2\theta.$$

q will be a maximum when $\sin 2\theta$ is a maximum, i.e. when $\theta = 45°$.

Only two stresses have been considered, p_x and p_y, but there is a possible third stress p_z in the third plane. Generally p_z is zero, so that the maximum shear stress $= \frac{1}{2}(p_x - 0) = \frac{1}{2}p_x$, if p_x is greater than p_y and of the same sign.

The resultant of p_n and q is given by

$$r^2 = p_n^2 + q^2$$
$$= (p_x\cos^2\theta + p_y\sin^2\theta)^2 + \{\tfrac{1}{2}(p_x - p_y)\sin 2\theta\}^2$$
$$= p_x^2\cos^4\theta + p_y^2\sin^4\theta + 2p_xp_y\sin^2\theta\cos^2\theta + (p_x - p_y)^2(\sin\theta\cos\theta)^2$$
$$= p_x^2\cos^4\theta + p_y^2\sin^4\theta + 2p_xp_y\sin^2\theta\cos^2\theta + p_x^2\sin^2\theta\cos^2\theta + p_y^2\sin^2\theta\cos^2\theta - 2p_xp_y\sin^2\theta\cos^2\theta$$
$$= p_x^2\cos^2\theta(\cos^2\theta + \sin^2\theta) + p_y^2\sin^2\theta(\sin^2\theta + \cos^2\theta)$$
$$= p_x^2\cos^2\theta + p_y^2\sin^2\theta.$$

$$\therefore\ r = \sqrt{(p_x^2\cos^2\theta + p_y^2\sin^2\theta)}.$$

The inclination of r to the plane AC is given by

$$\tan\alpha = \frac{p_n}{q} = \frac{p_x\cos^2\theta + p_y\sin^2\theta}{\frac{1}{2}(p_x - p_y)\sin 2\theta}$$
$$= \frac{p_x\cos^2\theta + p_y\sin^2\theta}{(p_x - p_y)\sin\theta\cos\theta}$$
$$= \frac{p_x\cot\theta + p_y\tan\theta}{p_x - p_y}.$$

Example 4.—Tensile stresses of 4 tons per sq. in. and 2 tons per sq. in. act on planes at right angles to one another. Find the normal, tangential and resultant stresses on planes making 0°, 30°, 45°, 60° 90°, 120°, 135°,

150° and 180° with the plane on which the 4 tons per sq. in. stress acts. Plot curves connecting stresses and angles.

Applying formulæ $p_n = p_x \cos^2\theta + p_y \sin^2\theta$,

$$q = \tfrac{1}{2}(p_x - p_y)\sin 2\theta,$$

$$r = \sqrt{(p_x^2 \cos^2\theta + p_y^2 \sin^2\theta)},$$

where p_x = 4 tons per sq. in., p_y = 2 tons per sq. in., we have

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
p_n	4	3·5	3·0	2·5	2	2·5	3·0	3·5	4
q	0	·866	1·0	·866	0	−·866	−1·0	−·866	0
r	4	3·61	3·16	2·65	2	2·65	3·16	3·61	4

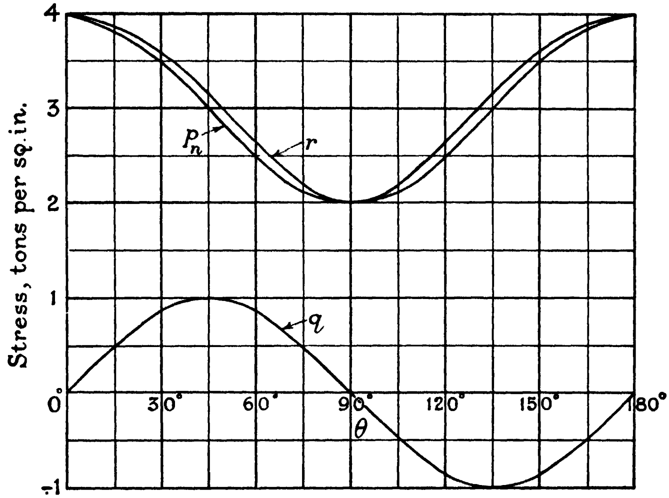


Fig. 8.6

(ii) *Unlike stresses.*—Let p_x be tensile and p_y compressive. It will not be necessary to deduce the expressions for p_n and q from first principles, as the method of derivation would be the same as for like stresses.

Substituting in the formulæ obtained for like stresses, but taking the compressive stress as negative,

$$p_n = p_x \cos^2\theta + (-p_y)\sin^2\theta$$
$$= p_x \cos^2\theta - p_y \sin^2\theta,$$
$$q = \tfrac{1}{2}[p_x - (-p_y)]\sin 2\theta$$
$$= \tfrac{1}{2}(p_x + p_y)\sin 2\theta.$$

The maximum shear stress will occur when $\theta = 45°$, and will be $= \frac{1}{2}(p_x + p_y)$.

4. Two Perpendicular Normal Stresses combined with a Shear Stress.

Let total stresses P_y and P_x act on planes AB and BC respectively, and total shear stresses F_{s_1} and F_{s_2} act on planes AB and BC respectively.

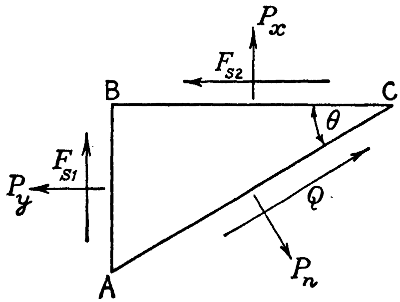


Fig. 8.7

If area of BC is A, then area of AB is $A \tan\theta$, and area of AC is $A \sec\theta$, then

$$p_x = \frac{P_x}{A},$$

$$p_y = \frac{P_y}{A \tan\theta},$$

$$f_s = \frac{F_{s_1}}{A \tan\theta} = \frac{F_{s_2}}{A}.$$

Normal stress on oblique plane $p_n = \dfrac{P_n}{A \sec\theta}$.

Shear stress on oblique plane $q = \dfrac{Q}{A \sec\theta}$.

Resolving the total stresses normal to AC,

$$P_n = P_x \cos\theta + P_y \sin\theta + F_{s_1} \cos\theta + F_{s_2} \sin\theta,$$

i.e. $p_n A \sec\theta = p_x A \cos\theta + p_y A \tan\theta \sin\theta + f_s A \tan\theta \cos\theta + f_s A \sin\theta$.

$$\therefore\ p_n = p_x \cos^2\theta + p_y \sin^2\theta + f_s \sin\theta\cos\theta + f_s \sin\theta\cos\theta$$

$$= p_x \cos^2\theta + p_y \sin^2\theta + f_s \sin 2\theta.$$

Resolving the total stresses parallel to AC, the algebraic sum must be zero for equilibrium of the section, i.e.

$$Q + P_x \sin\theta - P_y \cos\theta + F_{s_1} \sin\theta - F_{s_2} \cos\theta = 0.$$

$$\therefore\ Q = -P_x \sin\theta + P_y \cos\theta - F_{s_1} \sin\theta + F_{s_2} \cos\theta.$$

$$\therefore\ qA \sec\theta = -p_x A \sin\theta + p_y A \tan\theta \cos\theta - f_s A \tan\theta \sin\theta + f_s A \cos\theta.$$

$$\therefore\ q = -p_x \sin\theta \cos\theta + p_y \sin\theta \cos\theta - f_s \sin^2\theta + f_s \cos^2\theta$$

$$= \tfrac{1}{2}(p_y - p_x) \sin 2\theta + f_s \cos 2\theta.$$

The angle θ for maximum p_n is found by differentiating p_n with respect to θ and equating to zero, i.e.

$$\frac{dp_n}{d\theta} = -2p_x \cos\theta \sin\theta + 2p_y \cos\theta \sin\theta + 2f_s \cos 2\theta$$

$$= -2[\tfrac{1}{2}(p_x - p_y) \sin 2\theta - f_s \cos 2\theta]$$

$$= 0.$$

But $$q = \tfrac{1}{2}(p_y - p_x) \sin 2\theta + f_s \cos 2\theta,$$

therefore p_n is a maximum when $q = 0$.

Now $$q = 0 = -\tfrac{1}{2}(p_x - p_y) \sin 2\theta + f_s \cos 2\theta,$$

or $$\tan 2\theta = \frac{2f_s}{p_x - p_y}.$$

$$\therefore\ 2\theta = \tan^{-1}\left(\frac{2f_s}{p_x - p_y}\right),$$

or $$= \tan^{-1}\left(\frac{2f_s}{p_x - p_y}\right) + 180^\circ.$$

Hence there are two values of θ for which q is zero. The two values are separated by 90°. Any plane on which there is only a normal stress is known as a *principal plane*, and the normal stresses are known as *principal stresses*. Hence p_n has maximum values on two planes separated by 90°, and the planes are principal planes and the two stresses p_n are principal stresses.

5. Principal Stresses.

The values of the principal stresses may be found by obtaining the maximum values of p_n in the previous section, but if the definition for principal stresses and planes be used, it is possible to arrange the stresses on the prism so that there is no shear on plane AC. A com-

parable case has been treated in *Materials and Structures*, Chap. VII, by finding the maximum values of p_n, and the present problem could be solved in a similar manner.

Assume a principal total stress P_n acting on plane AC, and P_y, P_x, F_{s_1} and F_{s_2} as total normal and shear stresses on planes AB and BC respectively. The area of BC is A, so that the areas of AB and AC are $A \tan\theta$ and $A \sec\theta$ respectively.

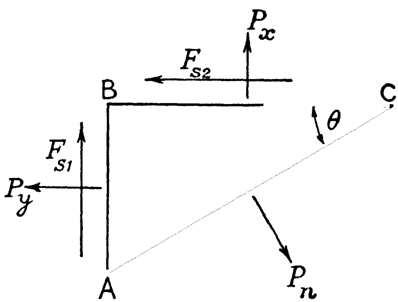


Fig. 8.8

Now
$$P_x = Ap_x,$$
$$P_y = A \tan\theta \,.\, p_y,$$
$$F_{s_1} = A \tan\theta \,.\, f_s,$$
$$F_{s_2} = Af_s,$$
and
$$P_n = A \sec\theta \,.\, p_n.$$

Resolving all the forces in the direction of P_x,

$$P_n \cos\theta = P_x + F_{s_1}.$$
$$p_n A \sec\theta \cos\theta = Ap_x + Af_s \tan\theta,$$
or
$$p_n = p_x + f_s \tan\theta.$$
$$\therefore \tan\theta = \frac{p_n - p_x}{f_s}. \quad \ldots \quad (a)$$

Resolving all the forces in direction of P_y,

$$P_n \sin\theta = P_y + F_{s_2}.$$
$$\therefore p_n A \sec\theta \sin\theta = Ap_y \tan\theta + Af_s,$$
$$\therefore p_n \tan\theta = p_y \tan\theta + f_s.$$
$$\therefore \tan\theta = \frac{f_s}{p_n - p_y}. \quad \ldots \quad (b)$$

Equating (*a*) and (*b*),
$$\frac{p_n - p_x}{f_s} = \frac{f_s}{p_n - p_y}.$$
$$\therefore (p_n - p_x)(p_n - p_y) = f_s^2,$$
$$\therefore p_n^2 - (p_x + p_y)p_n + p_x p_y = f_s^2.$$
$$\therefore p_n = \frac{(p_x + p_y) \pm \sqrt{[(p_x + p_y)^2 - 4(p_x p_y - f_s^2)]}}{2}.$$
$$\therefore p_n = \tfrac{1}{2}\{(p_x + p_y) \pm \sqrt{[(p_x - p_y)^2 + 4f_s^2]}\}.$$

When deducing expressions for p_n and q on an oblique plane where two perpendicular normal stresses acted, the normal stresses were in effect principal stresses, as no shear stresses occurred on the planes. The two planes on which p_x and p_y acted were perpendicular to one another. It was shown that under such conditions the maximum value of q was one half the difference between the two normal stresses. The two principal stresses obtained above are at right angles to one another, hence $q = \frac{1}{2}(p_{n_1} - p_{n_2})$, and the planes on which q is a maximum make 45° with the principal planes. Thus

$$p_{n_1} = \tfrac{1}{2}\{(p_x + p_y) + \sqrt{[(p_x - p_y)^2 + 4f_s^2]}\},$$

$$p_{n_2} = \tfrac{1}{2}\{(p_x + p_y) - \sqrt{[(p_x - p_y)^2 + 4f_s^2]}\},$$

$$q = \tfrac{1}{2}\sqrt{[(p_x - p_y)^2 + 4f_s^2]}.$$

The case where p_x or p_y is zero has been fully covered in *Materials and Structures*, although the same expressions can be deduced from the above equations by assuming p_x or p_y zero.

Let $p_y = 0$, then

$$p_n = \tfrac{1}{2}\{p_x \pm \sqrt{(p_x^2 + 4f_s^2)}\},$$

$$q = \tfrac{1}{2}\sqrt{(p_x^2 + 4f_s^2)}.$$

Example 5.—The stresses on a flat plate are two normal stresses on planes at right angles to one another together with shear stresses on the two planes. The normal stresses are 5 tons per sq. in. tensile, and 3 tons per sq. in. compressive, whilst the shear stresses are 3 tons per sq. in. Find the values of the principal stresses, the angle made by the principal planes to the plane on which the 5 tons per sq. in. stress acts, and the maximum shear stress in the material.

Applying formula

$$p_n = \tfrac{1}{2}\{(p_x + p_y) \pm \sqrt{[(p_x - p_y)^2 + 4f_s^2]}\},$$

where $p_x = 5$ tons per sq. in. tensile,

$p_y = 3$ tons per sq. in. compressive, i.e. $p_y = -3$,

$f_s = 3$ tons per sq. in.

$$\therefore\ p_n = \tfrac{1}{2}\{(5 - 3) \pm \sqrt{[(5 + 3)^2 + 4 \times 3^2]}\}$$

$$= \tfrac{1}{2}(2 \pm \sqrt{100})$$

$$= 6 \text{ or } -4 \text{ tons per sq. in.}$$

Applying formula $\tan\theta = \frac{p_n - p_x}{f_s}$,

when $p_n = 6$, $\tan\theta = \frac{6-5}{3}$

$= \frac{1}{3}$

$= \tan 18° 20'$;

when $p_n = -4$, $\tan\theta = \frac{-4-5}{3}$

$= -3$

$= \tan 108° 30'$.

Applying formula, $q = \frac{1}{2}\sqrt{[(p_x - p_y)^2 + 4f_s^2]}$

$= \frac{1}{2}\sqrt{[(5+3)^2 + 4 \times 3^2]}$

$= \frac{1}{2}\sqrt{100}$

$= 5$ tons per sq. in.

✓*Example* 6.—The bending stress in a beam is 6 tons per sq. in. at a point where the shear stress is 3 tons per sq. in. Find the magnitude of the principal stresses, and the angle the principal planes make with the plane on which the 6 tons per sq. in stress acts. What is the magnitude and direction of the maximum shear stress?

Since there is only one normal stress the equation

$$p_n = \frac{1}{2}\{(p_x \pm \sqrt{[p_x^2 + 4f_s^2]}\}$$

may be used. $p_x = 6$ tons per sq. in., and $f_s = 3$ tons per sq. in.

$$\therefore p_n = \frac{1}{2}\{6 \pm \sqrt{(6^2 + 4 \times 3^2)}\}$$

$$= \frac{1}{2}\{6 \pm 6\sqrt{2}\}$$

$$= \frac{1}{2}\{6 \pm 8{\cdot}5\}$$

$$= 7{\cdot}25 \text{ or } -1{\cdot}25 \text{ tons per sq. in.},$$

i.e. 7·25 tons per sq. in. tensile and 1·25 tons per sq. in. compressive.

Applying formula $\tan\theta = \frac{p_n - p_x}{f_s}$,

when $p_n = 7{\cdot}25$, $\tan\theta = \frac{7{\cdot}25 - 6}{3}$

$= {\cdot}4167$

$= \tan 22° 40'$;

when $p_n = -1{\cdot}25$, $\tan\theta = \dfrac{-1{\cdot}25 - 6}{3}$

$= -2{\cdot}417$

$= \tan 112^\circ\ 30'$.

Applying formula, $q = \frac{1}{2}\sqrt{(p_x^2 + 4f_s^2)}$

$= \frac{1}{2}\sqrt{72}$

$= 4{\cdot}25$ tons per sq. in.

For q to be a maximum the planes must make 45° with the principal planes. Hence the planes make 67° 30′ and 157° 20′ with the plane on which the 6 tons per sq. in. stress acts.

Example 7.—The longitudinal thrust on a propeller shaft sets up a compressive stress of 4 tons per sq. in., whilst the twisting moment induces a shear stress on the surface of 5 tons per sq. in. Calculate the principal stresses and the maximum shear stress.

Applying formula, $p_n = \frac{1}{2}\{p_x \pm \sqrt{(p_x^2 + 4f_s^2)}\}$,

where $p_x = -4$ tons per sq. in., $f_s = 5$ tons per sq. in.,

$$p_n = \tfrac{1}{2}\{-4 \pm \sqrt{[(-4)^2 + 4 \times 5^2]}\}$$
$$= \tfrac{1}{2}\{-4 \pm \sqrt{116}\}$$
$$= \tfrac{1}{2}\{-4 \pm 10{\cdot}77\}$$

$= 3{\cdot}39$ tons per sq. in. tensile, and

$7{\cdot}39$ tons per sq. in. compressive.

Applying formula, $q = \frac{1}{2}\sqrt{(p_x^2 + 4f_s^2)}$

$= \frac{1}{2}\sqrt{[(-4)^2 + 4 \times 5^2]}$

$= 5{\cdot}39$ tons per sq. in.

6. Strain on an Oblique Plane due to a Direct Strain on a Given Plane.

Let the rectangular body ABCD be strained so that it changes to the rectangular body ABEF, where the sides BC and AD have been stretched by an equal amount. Sides AB and CD have remained the same length, so that EF = CD, but the diagonal BD has been stretched.

The strain on side BC $= e_x$.

The increase in length of BC $=$ BC . e_x,

hence the length BE $=$ BC $+$ BC . e_x

$=$ BC$(1 + e_x)$.

$$\text{The strain on plane BD} = e.$$

$$\text{The increase in length of BD} = \text{BD} \,.\, e,$$

hence

$$\text{the length BF} = \text{BD} + \text{BD} \,.\, e$$
$$= \text{BD}(1 + e).$$

By geometry,

$$\text{BF}^2 = \text{BE}^2 + \text{EF}^2$$
$$= \text{BC}^2(1 + e_x)^2 + \text{EF}^2$$
$$= \text{BC}^2(1 + 2e_x) + \text{EF}^2,$$

when the term containing e_x^2 is neglected since the strain is small.

$$\therefore \text{BF}^2 = \text{BC}^2 + \text{EF}^2 + \text{BC}^2 \,.\, 2e_x$$
$$= \text{BC}^2 + \text{CD}^2 + \text{BC}^2 \,.\, 2e_x,$$

since EF = CD.

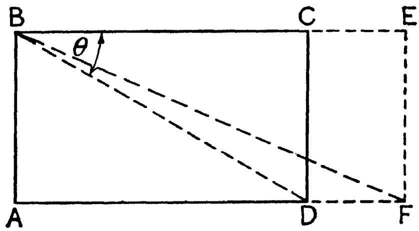


Fig. 8.9

But, from geometry,

$$\text{BD}^2 = \text{BC}^2 + \text{CD}^2.$$
$$\therefore \text{BF}^2 = \text{BD}^2 + \text{BC}^2 \,.\, 2e_x.$$

Now

$$\text{BF} = \text{BD}(1 + e),$$
$$\therefore \text{BD}^2(1 + e)^2 = \text{BD}^2 + \text{BC}^2 \,.\, 2e_x,$$
$$\therefore \text{BD}^2(1 + 2e) = \text{BD}^2 + \text{BC}^2 \,.\, 2e_x,$$

when the term in e^2 is neglected.

$$\therefore \text{BD}^2 \,.\, 2e = \text{BC}^2 \,.\, 2e_x.$$
$$\therefore e = \frac{\text{BC}^2}{\text{BD}^2} \,.\, e_x$$
$$= \cos^2\theta \,.\, e_x.$$

If the strain had taken place on sides AB and DC, and not on BC and AD, then the strain $e = \sin^2\theta \,.\, e_y$, where e_y is the strain on AB and DC.

7. Direct Strain on an Oblique Plane due to a Shear Strain on a Given Plane.

Let the rectangular body ABCD be strained so that it changes to parallelogram BCFE.

$$\text{The shear strain} = \phi.$$

Neglecting the difference between CG and DF, and between FG and CD, then

$$CG = FG \,.\, \phi.$$

$$\text{The strain on plane } BD = e.$$

$$\text{The increase in length } BD \text{ is } BD \,.\, e,$$

hence $\quad$ the length $BF = BD(1 + e)$.

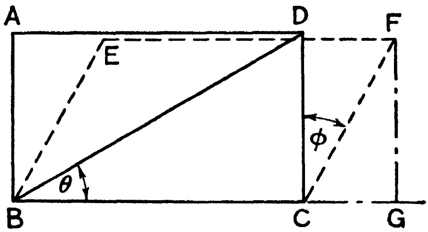


Fig. 8.10

$$\begin{aligned} \text{By geometry, } BF^2 &= BG^2 + FG^2 \\ &= (BC + CG)^2 + FG^2 \\ &= (BC + FG \,.\, \phi)^2 + FG^2 \\ &= BC^2 + 2BC \,.\, FG \,.\, \phi + FG^2, \end{aligned}$$

when the term $FG^2 \,.\, \phi^2$ is neglected.

$$\begin{aligned} \text{By geometry, } \quad BD^2 &= BC^2 + CD^2 \\ &= BC^2 + FG^2. \end{aligned}$$

$$\therefore\ BF^2 = BD^2 + 2BC \,.\, FG \,.\, \phi,$$

i.e. $\quad BD^2(1 + e)^2 = BD^2 + 2BC \,.\, FG \,.\, \phi,$

or $\quad BD^2 + 2BD^2 \,.\, e = BD^2 + 2BC \,.\, FG \,.\, \phi,$

when the term $BD^2 \,.\, e^2$ is neglected.

$$\therefore\ 2\mathrm{BD}^2 \,.\, e = 2\mathrm{BC} \,.\, \mathrm{FG} \,.\, \phi,$$

$$\therefore\ e = \frac{\mathrm{BC}}{\mathrm{BD}} \cdot \frac{\mathrm{FG}}{\mathrm{BD}} \cdot \phi$$

$$= \frac{\mathrm{BC}}{\mathrm{BD}} \cdot \frac{\mathrm{CD}}{\mathrm{BD}} \cdot \phi$$

$$= \cos\theta \,.\, \sin\theta \,.\, \phi$$

$$= \tfrac{1}{2}\,\phi \sin 2\theta.$$

8. General Case of Strain in Two Dimensions.

In this case, direct strains on two given planes at right angles to one another, and a shear strain, are assumed, and their effect on the direct strain on an oblique plane is required. It is a combination of the previous sections, so that

$$e = e_x \cos^2\theta + e_y \sin^2\theta + \tfrac{1}{2}\phi \sin 2\theta$$

$$= \tfrac{1}{2}e_x(\cos 2\theta + 1) + \tfrac{1}{2}e_y(1 - \cos 2\theta) + \tfrac{1}{2}\phi \sin 2\theta$$

$$= \tfrac{1}{2}[(e_x + e_y) + e_x \cos 2\theta - e_y \cos 2\theta + \phi \sin 2\theta]$$

$$= \tfrac{1}{2}[(e_x + e_y) + \cos 2\theta\,(e_x - e_y) + \phi \sin 2\theta].$$

For maximum e, $de/d\theta = 0$.

$$\therefore\ \frac{de}{d\theta} = -\sin 2\theta\,(e_x - e_y) + \phi \cos 2\theta = 0,$$

$$\therefore\ \tan 2\theta = \frac{\phi}{e_x - e_y}.$$

Now $$\sec^2 2\theta = \tan^2 2\theta + 1$$

$$= \left(\frac{\phi}{e_x - e_y}\right)^2 + 1$$

$$= \frac{\phi^2 + (e_x - e_y)^2}{(e_x - e_y)^2}.$$

$$\cos^2 2\theta = \frac{\qquad}{\phi^2 + (e_x - e_y)^2},$$

$$\therefore\ \cos 2\theta = \pm\sqrt{\frac{(e_x - e_y)^2}{\phi^2 + (e_x - e_y)^2}}.$$

Now $$e = \tfrac{1}{2}\{(e_x + e_y) + \cos 2\theta\,[(e_x - e_y) + \phi \tan 2\theta]\}.$$

Substituting for cos 2θ and tan 2θ,

$$e = \tfrac{1}{2}\left\{(e_x + e_y) \pm \sqrt{\left[\frac{(e_x - e_y)^2}{\phi^2 + (e_x - e_y)^2}\right] \cdot \left[(e_x - e_y) + \phi\left(\frac{\phi}{e_x - e_y}\right)\right]}\right\}$$

$$= \tfrac{1}{2}\left\{(e_x + e_y) \pm \sqrt{\left[\frac{(e_x - e_y)^2}{\phi^2 + (e_x - e_y)^2}\right] \cdot \left[\frac{(e_x - e_y)^2 + \phi^2}{(e_x - e_y)}\right]}\right\}$$

$$= \tfrac{1}{2}\{(e_x + e_y) \pm \sqrt{[(e_x - e_y)^2 + \phi^2]}\}.$$

The two strains given are the principal strains. They will be separated by 90°, since tan 2θ will have two values separated by 180°, hence the values of θ will be separated by 90°.

Example 8.—A block of material is subjected to a tensile strain and a compressive strain on planes at right angles to each other. In addition to the direct strains there is a shear strain of 4×10^{-6}. If the values of the tensile and compressive strains are 4×10^{-6} and 5×10^{-6} respectively, and there is no strain on planes at right angles to the above planes, find the principal strains in magnitude and direction.

Applying formula,

$$e = \tfrac{1}{2}\{(e_x + e_y) \pm \sqrt{[(e_x - e_y)^2 + \phi^2]}\},$$

where $e_x = 4 \times 10^{-6}$ tensile,

$e_y = 5 \times 10^{-6}$ compressive $= -5 \times 10^{-6}$,

$\phi = 4 \times 10^{-6}$,

$$e = \tfrac{1}{2}\{(4 \times 10^{-6} - 5 \times 10^{-6}) \pm \sqrt{[(4 \times 10^{-6} + 5 \times 10^{-6})^2 + (4 \times 10^{-6})^2]}\}$$

$$= \tfrac{1}{2} \times 10^{-6}[(-1) \pm \sqrt{(9^2 + 4^2)}]$$

$$= \tfrac{1}{2} \times 10^{-6}[-1 \pm 9{\cdot}85]$$

$= 4{\cdot}425 \times 10^{-6}$ tensile, and $5{\cdot}425 \times 10^{-6}$ compressive.

Applying formula,

$$\tan 2\theta = \frac{\phi}{e_x - e_y}$$

$$= \frac{4 \times 10^{-6}}{4 \times 10^{-6} + 5 \times 10^{-6}}$$

$$= \tfrac{4}{9}$$

$= \tan 24°$ and $\tan 204°$.

$\therefore\ \theta = 12°$ and $102°$.

9. Maximum Shear Strain.

Let the rectangular body ABCD be subjected to principal strains e_1 and e_2 on sides AD and CD respectively. If the strains are tensile,

the body will have increased in size to EFGH, and for simplicity it is shown as having increased in all directions. The centre O of rectangle ABCD is the centre of rectangle EFGH.

Produce AD to meet GH in J, and draw the diagonals of both rectangles.

$$\text{The strain on AD} = e_1 \,.\, \text{AD} = 2\text{DJ}.$$

$$\text{The strain on CD} = e_2 \,.\, \text{CD} = 2\text{JH}.$$

Now $$\text{AD} = 2\text{OD}\cos\theta, \text{ and CD} = 2\text{OD}\sin\theta,$$

$$\therefore\ \text{DJ} = e_1 \,.\, \text{OD}\cos\theta, \text{ and JH} = e_2 \,.\, \text{OD}\sin\theta.$$

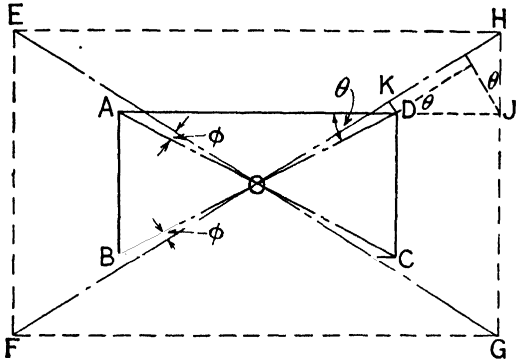


Fig. 8.11

The two diagonals will move through an angle ϕ when the strains are applied, so that the shear strain will be 2ϕ.

For point D the shear strain = DK/OD,

i.e. $$\phi = \frac{\text{DK}}{\text{OD}} = \frac{\text{DJ}\sin\theta - \text{JH}\cos\theta}{\text{OD}}$$

$$= \frac{e_1 \,.\, \text{OD}\cos\theta\sin\theta - e_2 \,.\, \text{OD}\sin\theta\cos\theta}{\text{OD}}$$

$$= e_1\cos\theta\sin\theta - e_2\sin\theta\cos\theta$$

$$= \tfrac{1}{2}(e_1 - e_2)\sin 2\theta.$$

But the net shear strain $= 2\phi = (e_1 - e_2)\sin 2\theta$.

The maximum shear strain will occur when $\sin 2\theta = 90°$, i.e. $\theta = 45°$.

The maximum shear strain equals the difference between the principal strains and occurs on a plane midway between the two principal planes.

Example 9.—Find the magnitude and direction of the maximum shear strain for the block of material in Example 8 when the strains stated occur.

Maximum shear strain $= (e_1 - e_2)$,

where $e_1 = 4{\cdot}425 \times 10^{-6}$ tensile,

and $e_2 \quad 5{\cdot}425 \times 10^{-6}$ compressive

$= -5{\cdot}425 \times 10^{-6}$.

$\therefore$ Maximum shear strain $= (4{\cdot}425 \times 10^{-6} + 5{\cdot}425 \times 10^{-6})$,

$= 9{\cdot}85 \times 10^{-6}$.

Angle of plane with side of rectangle

$= 24° + 45° = 69°$.

10. Principal Strains.

If three principal stresses p_1, p_2 and p_3 act on a block and they are all of the same type, then the total strain in the direction of the p_1 stress is

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE}$$

$$= \frac{p_1}{E} - \frac{p_2 + p_3}{mE},$$

where p_2/mE and p_3/mE are the lateral contractions produced by p_2 and p_3 in the direction of p_1.

Similarly $$e_2 = \frac{p_2}{E} - \frac{p_1 + p_3}{mE},$$

and $$e_3 = \frac{p_3}{E} - \frac{p_1 + p_2}{mE}.$$

For a two-dimensional stress condition, $p_3 = 0$ and $e_3 = 0$, and the difference between e_1 and e_2 is

$$(e_1 - e_2) = \left(\frac{p_1}{E} - \frac{p_2}{mE}\right) - \left(\frac{p_2}{E} - \frac{p_1}{mE}\right)$$

$$= \left(\frac{p_1}{E} - \frac{p_2}{E}\right) + \left(\frac{p_1}{mE} - \frac{p_2}{mE}\right)$$

$$= \frac{1}{E}(p_1 - p_2) + \frac{1}{mE}(p_1 - p_2)$$

$$= (p_1 - p_2)\left(\frac{1}{E} + \frac{1}{mE}\right)$$

$$= \frac{(p_1 - p_2)}{E}\left(\frac{m+1}{m}\right).$$

In Section 9 of this chapter the relation between the maximum shear strain and the principal strains was established. It will be useful to check the result by stress considerations. The maximum shear stress q is one half the difference between the principal stresses, i.e.

$$q = \tfrac{1}{2}(p_1 - p_2).$$

$$\therefore\ (e_1 - e_2) = \frac{2q}{E}\left(\frac{m+1}{m}\right).$$

But from the relationship between the elastic constants,

$$\frac{1}{C} = \frac{2}{E}\left(1 + \frac{1}{m}\right),$$

$$\therefore\ \frac{1}{E} = \frac{1}{2C\left(1 + \frac{1}{m}\right)}$$

$$= \frac{m}{2C(m+1)}.$$

Hence $$e_1 - e_2 = \frac{2qm}{2C(m+1)} \cdot \frac{m+1}{m}$$

$$= \frac{q}{C}.$$

But $$\frac{q}{C} = \text{shear strain},$$

$$\therefore\ e_1 - e_2 = \text{shear strain}.$$

11. Resistance to Lateral Expansion.

(i) Let the material be subjected to a longitudinal stress p_1, but, whilst free to contract laterally in the direction e_3, let it be prevented from contracting in the direction e_2.

Then $e_2 = 0$ and $p_3 = 0$,

$$\therefore\ e_1 = \frac{p_1}{E} - \frac{p_2}{mE}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$e_2 = 0 = \frac{p_2}{E} - \frac{p_1}{mE}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$e_3 = -\frac{p_1}{mE} - \frac{p_2}{mE}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

From (*b*), $$p_2 = \frac{p_1}{m}.$$

Substituting for p_2 in (*a*) and (*c*),

$$e_1 = \frac{1}{E}\left(p_1 - \frac{p_1}{m^2}\right)$$

$$= \frac{p_1}{E}\left(\frac{m^2 - 1}{m^2}\right); \quad \ldots \ldots \ldots (d)$$

$$e_3 = -\frac{p_1}{mE} - \frac{p_1}{m^2 E}$$

$$= -\frac{p_1}{mE}\left(1 + \frac{1}{m}\right). \quad \ldots \ldots (e)$$

Substituting for p_1 from (*d*) in (*e*),

$$e_3 = -\frac{e_1 E m^2}{(m^2 - 1)mE}\left(1 + \frac{1}{m}\right)$$

$$= -\frac{e_1 m}{m^2 - 1}\left(\frac{m + 1}{m}\right)$$

$$= -\frac{e_1}{m - 1}. \quad \ldots \ldots \ldots \ldots (f)$$

From (*d*), $$\frac{p_1}{e_1} = E\left(\frac{m^2}{m^2 - 1}\right).$$

But under free conditions $p_1/e_1 = E$, hence by preventing lateral contraction the effective modulus of elasticity has been raised to $E\left(\frac{m^2}{m^2 - 1}\right)$.

From (*f*), $$-\frac{e_3}{e_1} = \frac{1}{m - 1}.$$

But under free conditions $\frac{e_3}{e_1}$ = Poisson's ratio $\left(\frac{1}{m}\right)$, therefore, when lateral contraction is prevented, the effective Poisson's ratio has been increased from $1/m$ to $1/(m - 1)$.

(ii) Let the lateral contraction be prevented in both directions, so that

$$e_2 = e_3 = 0,$$

then $$p_2 = p_3.$$

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE}$$

$$= \frac{p_1}{E} - \frac{2p_2}{mE}. \quad \text{(i)}$$

$$e_2 = 0 = \frac{p_2}{E} - \frac{p_1}{mE} - \frac{p_3}{mE}$$

$$= \frac{p_2}{E}\left(1 - \frac{1}{m}\right) - \frac{p_1}{mE}. \quad \text{(ii)}$$

$$e_3 = 0 = \frac{p_3}{E} - \frac{p_1}{mE} - \frac{p_2}{mE}$$

$$= \frac{p_2}{E}\left(1 - \frac{1}{m}\right) - \frac{p_1}{mE}. \quad \text{(iii)}$$

From (ii) and (iii), $p_2 = p_3 = p_1 \cdot \dfrac{1}{m\left(1 - \dfrac{1}{m}\right)}$

$$= \frac{p_1}{m - 1}. \quad \text{(iv)}$$

Substituting for p_2 in (i),

$$e_1 = \frac{p_1}{E} - \frac{2p_1}{m(m-1)E}.$$

$$\therefore\ e_1 = \frac{p_1}{E}\left\{1 - \frac{2}{m(m-1)}\right\}$$

$$= \frac{p_1}{E}\left\{\frac{m^2 - m - 2}{m^2 - m}\right\}$$

$$= \frac{p_1}{E}\,\frac{(m-2)(m+1)}{m(m-1)}.$$

$$\therefore\ \frac{p_1}{e_1} = \frac{Em(m-1)}{(m-2)(m+1)}.$$

Under free expansion $p_1/e_1 = E$, therefore the effective modulus of elasticity has been raised to

$$\frac{Em(m-1)}{(m-2)(m+1)}.$$

As no lateral contraction occurs the effective Poisson's ratio is zero.

In dealing with the bending of flat plates the foregoing conditions arise in which lateral contraction is prevented.

12. Strain Energy due to Compound Stress.

For a cube of unit sides on which there are three pairs of tensile stresses p_1, p_2 and p_3, the net strain on the sides will be

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE},$$

$$e_2 = \frac{p_2}{E} - \frac{p_1 + p_3}{mE},$$

$$e_3 = \frac{p_3}{E} - \frac{p_1 + p_2}{mE}.$$

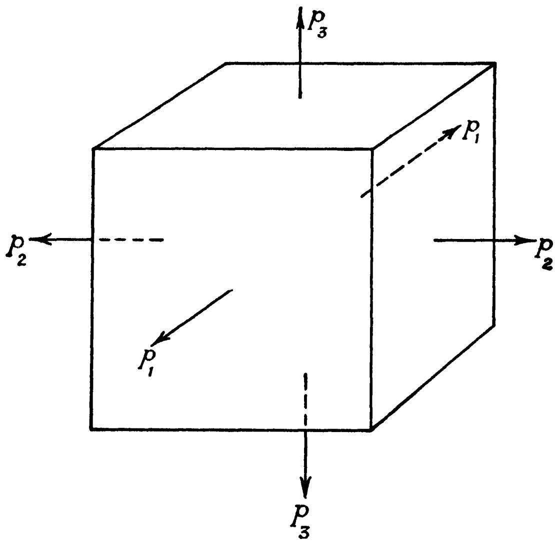


Fig. 8 12

Now strain-energy = average total stress × extension

= stress × strain

in this case as a unit cube is considered.

$$\text{Total strain-energy } U = \tfrac{1}{2}p_1e_1 + \tfrac{1}{2}p_2e_2 + \tfrac{1}{2}p_3e_3$$

$$= \tfrac{1}{2}p_1\left(\frac{p_1}{E} - \frac{p_2 + p_3}{mE}\right) + \tfrac{1}{2}p_2\left(\frac{p_2}{E} - \frac{p_1 + p_3}{mE}\right) + \tfrac{1}{2}p_3\left(\frac{p_3}{E} - \frac{p_1 + p_2}{mE}\right)$$

$$= \frac{1}{2E}\left\{(p_1^2 + p_2^2 + p_3^2) - \frac{2}{m}(p_1p_2 + p_1p_3 + p_2p_3)\right\}.$$

If it is a two-dimensional stress condition,

$$p_3 = 0, \text{ and } U = \frac{1}{2E}\left\{(p_1^2 + p_2^2) - \frac{2}{m}(p_1 p_2)\right\}.$$

EXAMPLES

1. Deduce the relation between the modulus of elasticity, the modulus of rigidity and Poisson's ratio.

A hollow shaft of 3 in. external diameter and 2 in. internal diameter twists through an angle of 0·6 degree in a length of 4 ft. when subjected to an axial twisting moment of 4 tons-in. Estimate the deflection at the centre of the shaft due to its own weight when placed in a horizontal position on supports 4 ft. apart. Weight of shaft = 13·2 lb. per ft. length. Poisson's ratio = 0·3. [0·0014 in.]

2. Obtain an expression for the principal stresses in a general two-dimensional system of stress.

At a certain point in a beam the bending moment gives rise to a longitudinal tensile stress of 4 tons per sq. in. and the shearing force gives rise to a shear stress of 4 tons per sq. in. on vertical and horizontal planes. Find the value of the greatest shear stress in the material at this point. [4·47 tons per sq. in.]

3. Prove that in a two-dimensional stress system the maximum shearing stress at a point is $\frac{1}{2}\sqrt{[(p_1 - p_2)^2 + 4q^2]}$, where p_1 and p_2 are the normal stresses and q is the shear stress acting on perpendicular planes through the point.

A solid circular shaft is subjected to a torque of T tons-in. and to a bending moment of M tons-in. The greatest principal stress is 5 tons per sq. in. The maximum shearing stress is 4 tons per sq. in. Find the ratio of M to T. [1 : 3·87.]

4. A beam of rectangular cross-section, depth d and breadth b, is subjected at a certain cross-section to a bending moment M and a transverse shearing force F in a direction parallel to the longer side d. Obtain expressions for the principal stresses and the maximum shear stress at a point in the cross-section at distance y from the neutral axis.

5. Direct stresses of 8 tons per sq. in. tension and 6 tons per sq. in. compression are applied to an elastic material at a certain point, on planes at right angles to one another. The greater principal stress in the material is limited to 10 tons per sq. in. tension. To what shearing stress may the material be subjected on the given planes, and what will then be the maximum shear stress at the point? Work from first principles. (London Univ.) [5·65 tons per sq. in.; 9 tons per sq. in.]

6. In a circular shaft subject to bending and twisting moments the greater principal stress is numerically five times the lesser one. Find the ratio of the bending moment to the twisting moment. [0·89.]

7. (*a*) Obtain relations between the elastic constants, modulus of elasticity E, modulus of rigidity C and Poisson's ratio $1/m$.

(*b*) In the experimental determination of E and C, if E is assumed to be correct and C is subject to an error of 3 per cent on either side of the mean value, find the fluctuation in the true value of Poisson's ratio which is $\frac{1}{4}$. [·288, ·214.]

8. (*a*) Define principal planes and principal stresses.

(*b*) At a certain point in a structural member there are stresses as follows: a tensile stress of 5 tons per sq. in., a compressive stress of 4 tons per sq. in. at right angles to the tensile stress, and a shear stress of 3 tons per sq. in. on each of the planes on which act the tensile and compressive stresses. Calculate the values of the principal stresses at this point, and the angles made with the horizontal by the planes on which the principal stresses act. Work from first principles or derive any formula used.

[4·9 tons per sq. in. compression; 5·9 tons per sq. in. tension; 16° 51′; 106° 51′.]

9. Explain the terms Poisson's ratio, Young's modulus and modulus of rigidity.

The intensities of three principal stresses in a boiler plate are, at a certain point, 8 tons per sq. in. tension, 4 tons per sq. in tension, and 0. Find what stress acting alone would produce the same strain in the direction of the 8 tons per sq. in. stress, given that the ratio of Young's modulus to modulus of rigidity 5 : 2. [7 tons per sq. in.]

10. At a point in a plate girder the intensity of vertical shear stress is 4 tons per sq. in. and that of a co-existent horizontal tensile stress is 6 tons per sq. in. From first principles find the values of principal stresses and the angles the principal planes make with the vertical.

What is the value of the maximum shear stress at the point?

[8 tons per sq. in. tension; 2 tons per sq. in. compression; 5 tons per sq. in. shear; 26° 34′ and −63° 26′ with the vertical.]

CHAPTER IX

Thick Cylinders subjected to Fluid Pressure

The radial stress and circumferential stress in a thick cylinder are expressed by the relationships

$$f_2 = a - \frac{b}{r^2}$$

and

$$p = \frac{b}{r^2} + a$$

respectively (see *Materials and Structures*, p. 160). These relationships were obtained by assuming that the radial stress at radius r was p and at $r + dr$ it was $p + dp$. It is advisable to demonstrate that the same relationships are obtained when $p + dp$ is the radial pressure at radius r.

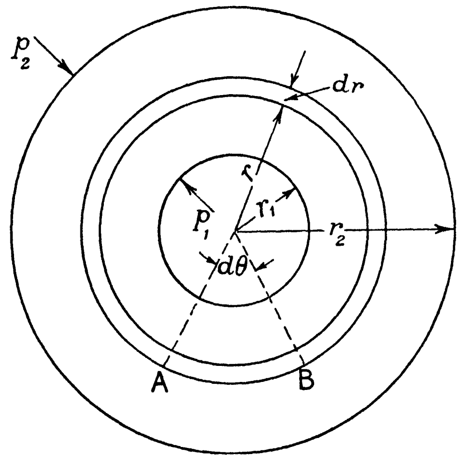


Fig. 9.1

Let r_1 and r_2 be the internal and external radii of the cylinder,

p_1 and p_2 the internal and external pressure (intensities),

p_y the circumferential stress on a ring of material of radius r and thickness dr,

$p_x + dp_x$ the internal pressure (compressive) on the ring, and

p_x the external pressure (compressive) on the ring.

The stresses on a small portion ABCD of the ring are shown in fig. 9.2.

Internal radial force on ABCD $= (p_x + dp_x)r\,d\theta$ per unit length.

External radial force on ABCD $= p_x(r + dr)\,d\theta$ per unit length.

Circumferential total stresses $= p_y\,dr$ per unit length.

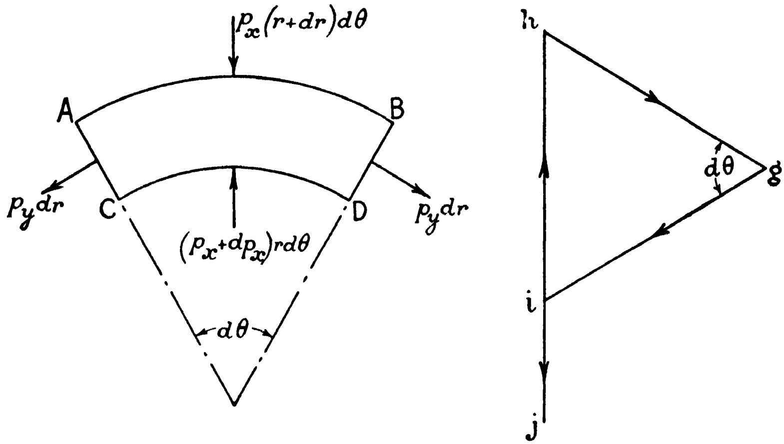


FIG. 9.2

Since this portion is in equilibrium under the action of these forces, a force polygon may be drawn and is as shown in fig. 9.2.

$$\text{gi} = p_y\,dr.$$

$$\text{hg} = p_y\,dr.$$

$$\text{ij} = p_x(r + dr)\,d\theta.$$

$$\text{jh} = (p_x + dp_x)r\,d\theta.$$

Then

$$\text{ih} = (p_x + dp_x)r\,d\theta - p_x(r + dr)\,d\theta$$

$$= dp_x\,r\,d\theta - p_x\,dr\,d\theta.$$

Since $d\theta$ is small,
$$\text{ih} = (\text{gi})\,d\theta$$
$$= p_y\,dr\,d\theta.$$
$$\therefore\ dp_x\,r\,d\theta - p_x\,dr\,d\theta = p_y\,dr\,d\theta,$$
or
$$dp_x\,r - p_x\,dr = p_y\,dr.$$
$$\therefore\ p_y = r\frac{dp_x}{dr} - p_x.$$

As there are three variables in this equation it will not be possible to solve unless we find a further relationship between two of the variables. It is possible to find such a relationship between p_x and p_y from consideration of the strain effects. The radial stresses p_x and $(p_x + dp_x)$ are tending to thin down the dimension dr and increase the length perpendicular to the plane of the paper, i.e. the lateral strain produced by the radial stress is $p_x/(mE)$, where $1/m$ is Poisson's ratio, and E is the modulus of elasticity.

The circumferential stress p_y is tensile and it is producing a lateral shortening perpendicular to the plane of the paper of $p_y/(mE)$.

There will be a longitudinal stress f, which may be assumed to be constant over the section and which produces a longitudinal strain of f/E. Hence the net longitudinal strain is

$$\frac{f}{E} + \frac{p_x}{mE} - \frac{p_y}{mE}.$$

If we assume that plane transverse sections perpendicular to the axis remain plane and perpendicular, and that all transverse sections are distorted by the same amount, then

$$\frac{f}{E} + \frac{p_x}{mE} - \frac{p_y}{mE}$$

is constant. But f, E and $1/m$ are constant, therefore $(p_x - p_y)$ must be constant.

Let
$$p_x - p_y = 2a,$$

where a is a constant.

Now
$$\frac{dp_x}{dr} = \frac{p_y + p_x}{r}$$
$$= \frac{2p_x - 2a}{r}.$$

But the inside pressure is greater than the outside pressure, so that the rate of change of pressure with respect to the radius is of negative slope. Hence

$$\frac{2p_x - 2a}{r} = -\frac{dp_x}{dr}.$$

Integrating, we get $$r^2 p_x = ar^2 + b,$$

where b is a constant, or

$$p_x = a + \frac{b}{r^2}.$$

$$\therefore\ p_y = -2a + \left(a + \frac{b}{r^2}\right)$$

$$= -a + \frac{b}{r^2}.$$

Example 1.—Two thick steel cylinders A and B, closed at the ends, have the same dimensions, the outer diameter of each being 1·5 times the inner diameter. The cylinder A is subject to internal fluid pressure only and B to external fluid pressure only. Find the ratio of the pressures on these cylinders when the greatest circumferential stress has the same value for each cylinder.

Applying the formulæ,

$$p_x = a + \frac{b}{r^2},$$

and $$p_y = \frac{b}{r^2} - a.$$

For A, $$p_x = p_1 \text{ when } r = r_1,$$

$$= 0 \text{ when } r = 1{\cdot}5r_1.$$

For B, $$p_x = p_2 \text{ when } r = 1{\cdot}5r_1,$$

$$= 0 \text{ when } r = r_1.$$

For equal circumferential maximum stress,

$$p_y = \frac{b_1}{r_1^2} - a_1 = \frac{b_2}{r_1^2} - a_2.$$

For A, $$p_1 = \frac{b_1}{r_1^2} + a_1,$$

$$0 = \frac{b_1}{(1{\cdot}5r_1)^2} + a_1.$$

By subtraction, $$p_1 = \frac{b_1}{r_1^2} - \frac{b_1}{(1{\cdot}5r_1)^2}.$$

$$\therefore\ b_1 = \frac{2{\cdot}25}{1{\cdot}25} p_1 r_1^2.$$

Now $$a_1 = -\frac{b_1}{(1{\cdot}5r_1)^2}$$

$$= -\frac{p_1}{1{\cdot}25}.$$

$$\therefore\ p_y = \frac{2{\cdot}25}{1{\cdot}25}\frac{p_1 r_1^2}{r_1^2} + \frac{p_1}{1{\cdot}25}$$

$$= \frac{3{\cdot}25}{1{\cdot}25} p_1. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

For B, $$p_2 = \frac{b_2}{(1{\cdot}5r_1)^2} + a_2,$$

$$0 = \frac{b_2}{r_1^2} + a_2.$$

By subtraction, $$p_2 = \frac{b_2}{(1{\cdot}5r_1)^2} - \frac{b_2}{r_1^2}.$$

$$\therefore\ b_2 = -\frac{2{\cdot}25}{1{\cdot}25} p_2 r_1^2,$$

and $$a_2 = \frac{2{\cdot}25}{1{\cdot}25} p_2.$$

Hence $$p_y = -\frac{2{\cdot}25}{1{\cdot}25}\frac{p_2 r_1^2}{r_1^2} - \frac{2{\cdot}25}{1{\cdot}25} p_2$$

$$= -\frac{4{\cdot}5}{1{\cdot}25} p_2. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

Equating (1) and (2), but changing the sign of (2) because stresses are of different types,

$$\frac{3{\cdot}25}{1{\cdot}25} p_1 = \frac{4{\cdot}5}{1{\cdot}25} p_2,$$

$$\therefore\ 3{\cdot}25 p_1 = 4{\cdot}5\, p_2,$$

$$\therefore\ p_1 : p_2 = 4{\cdot}5 : 3{\cdot}25.$$

1. Zero Internal Pressure.

The equations for an external pressure are

$$p_x = \frac{b}{r^2} + a,$$

$$p_y = a - \frac{b}{r^2}$$

(see *Materials and Structures*, Chap. VIII).

When the internal pressure is negligible the following conditions are applicable:

$$p_x = p_2 \text{ when } r = r_2,$$

$$= 0 \text{ when } r = r_1.$$

$$\therefore\ p_x = p_2 = \frac{b}{r_2^2} + a,$$

$$0 = \frac{b}{r_1^2} + a.$$

By subtraction,

$$p_2 = \frac{b}{r_2^2} - \frac{b}{r_1^2}.$$

$$\therefore\ b = \frac{p_2 r_1^2 r_2^2}{r_1^2 - r_2^2},$$

and

$$a = -\frac{p_2 r_2^2}{r_1^2 - r_2^2}.$$

$$\therefore\ p_x = \frac{p_2 r_1^2 r_2^2}{(r_1^2 - r_2^2)r^2} - \frac{p_2 r_2^2}{r_1^2 - r_2^2}$$

$$= \frac{p_2 r_2^2}{r_1^2 - r_2^2}\left[\frac{r_1^2}{r^2} - 1\right]$$

$$= \frac{p_2 r_2^2}{r_1^2 - r_2^2}\left[\frac{r_1^2 - r^2}{r^2}\right],$$

and

$$p_y = -\frac{p_2 r_2^2}{r_1^2 - r_2^2}\left[\frac{r^2 + r_1^2}{r^2}\right].$$

2. Zero External Pressure.

When the external pressure is atmospheric or negligible compared with the internal pressure*, using the equations on p. 192 we have the following conditions:

$$p_x = p_1 \text{ when } r = r_1,$$

$$p_x = 0 \text{ when } r = r_2.$$

$$\therefore\ p_1 = a + \frac{b}{r_1^2}, \qquad (1)$$

$$0 = a + \frac{b}{r_2^2}. \qquad (2)$$

Subtracting (2) from (1),

$$p_1 = \frac{b}{r_1^2} - \frac{b}{r_2^2}.$$

$$\therefore\ b = \frac{p_1 r_1^2 r_2^2}{r_2^2 - r_1^2},$$

and

$$a = -\frac{p_1 r_1^2}{r_2^2 - r_1^2}.$$

$$\therefore\ p_x = \frac{p_1 r_1^2 r_2^2}{(r_2^2 - r_1^2)r^2} - \frac{p_1 r_1^2}{r_2^2 - r_1^2}$$

$$= \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r^2} - 1\right)$$

$$= \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left(\frac{r_2^2 - r^2}{r^2}\right),$$

and

$$p_y = \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left(\frac{r_2^2 + r^2}{r^2}\right).$$

The maximum values of p_x and p_y will occur when $r = r_1$, and their values are

$$p_x = \frac{p_1}{r_2^2 - r_1^2}(r_2^2 - r_1^2) = p_1,$$

and

$$p_y = \frac{p_1}{r_2^2 - r_1^2}(r_2^2 + r_1^2).$$

If the ratio $r_2/r_1 = k$, then

$$p_y = p_1 \frac{k^2 + 1}{k^2 - 1}.$$

* It should be noted that p_y is a tensile stress and therefore the expression for p_y is of opposite sign to that obtained in Chapter VIII of *Materials and Structures*.

(i) *Maximum-principal-stress theory.*—The limiting stress is the stress at the limit of proportionality. When the principal stress reaches that value then failure occurs; consequently $p_y = f =$ limit of proportionality stress, i.e.

$$f = p_1 \frac{k^2+1}{k^2-1}.$$

(ii) *Maximum-shear-stress theory.*—The limiting factor is considered to be the greatest shear stress in the material. It has been shown in Chap. VIII, **3**, that the maximum shear stress equals one half the difference between the principal stresses. The principal stresses in the case of thick cylinders are p_1 and p_y, and are of opposite types, i.e. one is tensile whilst the other is compressive. Hence the maximum shear stress

$$= \tfrac{1}{2}(p_1 + p_y)$$

$$= \tfrac{1}{2}\left(p_1 + p_1 \frac{k^2+1}{k^2-1}\right)$$

$$= \tfrac{1}{2}\left(p_1 \frac{2k^2}{k^2-1}\right).$$

The maximum shear stress at the limit of proportionality in a tensile test equals one half the tensile stress at the limit of proportionality, i.e. $\tfrac{1}{2}f$. Hence

$$\tfrac{1}{2}f = \tfrac{1}{2}p_1 \frac{2k^2}{k^2-1},$$

$$\therefore f = p_1 \frac{2k^2}{k^2-1}.$$

(iii) *Maximum-strain theory.*—The greatest principal extension must be within the limits shown to be safe by the normal tensile test, i.e. not greater than the strain produced by the stress at the limit of proportionality. Without a longitudinal stress the greatest principal strain

$$= \frac{1}{E}\left(p_y + \frac{p_1}{m}\right)$$

$$= \frac{1}{E}\left(p_1 \frac{k^2+1}{k^2-1} + \frac{p_1}{m}\right)$$

$$= \frac{p_1}{E}\left(\frac{k^2+1+\frac{1}{m}(k^2-1)}{k^2-1}\right)$$

$$= \frac{p_1}{E}\left(\frac{k^2\left[1+\frac{1}{m}\right]+\left[1-\frac{1}{m}\right]}{k^2-1}\right).$$

The limiting strain $= \dfrac{f}{E}$.

$$\therefore \frac{f}{E} = \frac{p_1}{E}\left(\frac{k^2\left[1+\frac{1}{m}\right]+\left[1-\frac{1}{m}\right]}{k^2-1}\right),$$

i.e.
$$f = p_1\left(\frac{k^2\left[1+\frac{1}{m}\right]+\left[1-\frac{1}{m}\right]}{k^2-1}\right).$$

(iv) *Maximum-strain-energy theory.*—The resilience or capacity to store strain energy is the limiting factor. The maximum allowable resilience is that energy stored on stressing up to the limit of proportionality in a tensile test. Its value is given by $\frac{1}{2}fl$, where l is the extension produced by stress f. The strain energy per unit volume for a two-dimensional case in which the stresses are of opposite types is

$$\frac{1}{2E}\left\{p_1^2 + p_y^2 + \frac{2}{m}(p_1 p_y)\right\}.$$

But
$$\tfrac{1}{2}fl = \tfrac{1}{2}\frac{f^2}{E} \text{ per unit volume,}$$

hence
$$\tfrac{1}{2}\frac{f^2}{E} = \frac{1}{2E}\left\{p_1^2 + p_y^2 + \frac{2}{m}(p_1 p_y)\right\},$$

or
$$f^2 = p_1^2 + p_1^2\left(\frac{k^2+1}{k^2-1}\right)^2 + \frac{2}{m}p_1^2\left(\frac{k^2+1}{k^2-1}\right)$$

$$= p_1^2\left\{1 + \left(\frac{k^2+1}{k^2-1}\right)^2 + \frac{2}{m}\left(\frac{k^2+1}{k^2-1}\right)\right\}$$

$$= p_1^2\left\{\frac{2(k^4+1)+\frac{2}{m}(k^4-1)}{(k^2-1)^2}\right\}$$

$$= \frac{p_1^2}{(k^2-1)^2}\left\{\frac{2k^4(m+1)+2(m-1)}{m}\right\}.$$

$$\therefore f = \frac{p_1}{k^2-1}\sqrt{\left\{\frac{2k^4(m+1)+2(m-1)}{m}\right\}}.$$

Cook and Robertson carried out exhaustive tests on thick cylinders of both mild steel and cast iron. Their experimental results show that the maximum-principal-stress theory gives results which most nearly coincide with experimental results in the case of the cast-iron tubes—the stress f being taken as the maximum stress and not the stress at the

limit of proportionality. The mild-steel tubes gave results which fitted the maximum-strain-energy theory. Figs. 9.3 and 9.4 show the results for cast iron and mild steel respectively as derived from the above formula. Cook and Robertson's experimental results are shown for confirmation of the above deductions.

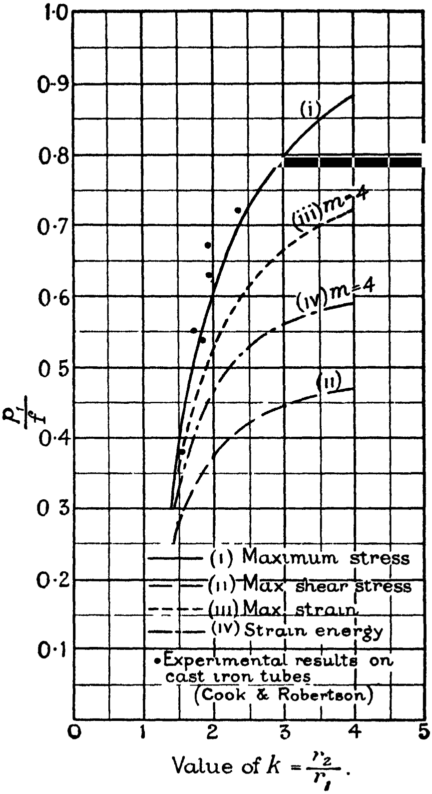


Fig. 9.3

3. Stress Distribution.

It is seen from the equation $p_y = \frac{b}{r^2} - a$ that the stress in the wall of the cylinder varies considerably where the thickness is great compared with the radius. There are three methods adopted for achieving a more uniform stress distribution:

(1) The cylinder is made by shrinking one tube on the outside of another. By increasing the number of tubes used to build up the cylinder a more uniform stress distribution is obtained.

(2) Wire is wrapped around the cylinder and tightened so that the wire is in tension; in consequence, the cylinder is in compression.

(3) The material of the cylinder is pre-stressed. A temporary internal pressure is applied to the cylinder so that plastic flow occurs in the material. As the stress distribution will not be uniform, the

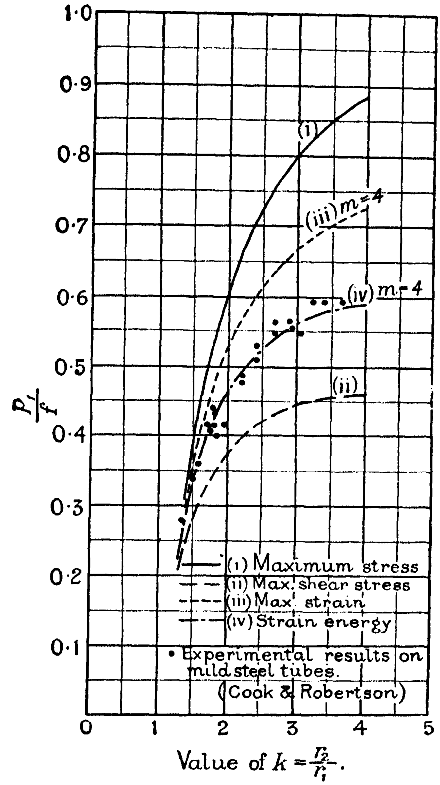


Fig. 9.4

amount of plastic flow will not be uniform throughout the wall thickness. When the pressure is removed, the material will not return to its original dimensions, since plastic flow has occurred. The stress-strain diagram on unloading will be a straight line, but, as the greatest amount of plastic flow will have occurred on the inside wall, a compressive stress will build up there, as it cannot return to its original dimensions.

Whichever of the above three methods is used, all are based on a compressive stress being induced in the material on the inside wall. When the actual internal pressure is applied, this compressive stress

must be balanced before any tensile stress is induced in the material. The problems of compound cylinders and wire-wound cylinders can be solved by using the thick-cylinder equations. The appropriate conditions must be used when finding the values of the constants a and b (see § **6** for wire-wound cylinders).

Example 2.- A cylinder of 6 in. internal diameter and 10 in. external diameter is formed by shrinking one tube of 8 in. internal diameter and 10 in. external diameter on to another tube with an external and internal diameter of 8 in. and 6 in. Find the radial compressive stress between the cylinders when the maximum tensile stress induced in the outer cylinder is 5 tons per sq. in.

Apply the formulæ $$p_x = a + \frac{b}{r^2},$$
$$p_y = \frac{b}{r^2} - a.$$

For the outer cylinder, $p_x = 0$ when $r = 5$ in.,

$p_y = 5$ tons per sq. in. when $r = 4$ in.,

$$\therefore 0 = a + \tfrac{1}{25}b,$$
$$5 = \tfrac{1}{16}b - a.$$

By addition, $$5 = \tfrac{1}{25}b + \tfrac{1}{16}b,$$
$$\therefore b = 48{\cdot}8.$$

Hence $$a = -48{\cdot}8/25$$
$$= -1{\cdot}95.$$

The radial pressure at the junction is given by

$$p_x = a + \frac{b}{r^2}, \text{ where } r = 4 \text{ in.}$$

Hence $$p_x = -1{\cdot}95 + (48{\cdot}8 \div 16)$$
$$= -1{\cdot}95 + 3{\cdot}05$$
$$= 1{\cdot}10 \text{ tons per sq. in.}$$

Example 3. - A compound cylinder is made by shrinking one tube on another, the final dimensions being: external diameter, 10 in.; internal diameter, 6 in.; diameter at junction of tubes, 9 in. If the radial pressure on the external surface of the inner tube at the common ($4\frac{1}{2}$ in.) radius is 4000 lb. per sq. in., determine the maximum tensile stress in the metal of the outer tube. (London Univ.)

Apply the formulæ $$p_x = a + \frac{b}{r^2},$$
$$p_y = \frac{b}{r^2} - a.$$

For the outer cylinder,

$$p_x = +4000 \text{ lb. per sq. in. when } r = 4\tfrac{1}{2} \text{ in.,}$$

since the external pressure on the inner cylinder equals the internal pressure on the outer cylinder. The equation $p_x = a + b/r^2$ was deduced for an internal pressure, and $p_x = 0$ when $r = 5$ in.,

$$\therefore\ 0 = a + b \div 5^2,$$

$$\therefore\ 4000 = a + b \div (4\tfrac{1}{2})^2.$$

By subtraction,

$$4000 = \frac{b}{(4\frac{1}{2})^2} - \frac{b}{5^2}.$$

$$\therefore\ b = \frac{4000 \times 25 \times 81}{19}$$

$$= 426{,}000,$$

and

$$a = -\frac{426{,}000}{25}$$

$$= -17{,}040.$$

The circumferential stress is given by

$$p_y = \frac{b}{r^2} - a,$$

and is a maximum when $r = 4\frac{1}{2}$ in.

$$\therefore\ p_y = \frac{426{,}000}{(4\frac{1}{2})^2} - (-17{,}040)$$

$$= 21{,}040 + 17{,}040$$

$$= 38{,}080 \text{ lb. per sq. in.}$$

Example 4.—A compound tube, 4 in. internal diameter and 8 in. external diameter, is made by shrinking one tube on to another. After cooling, a radial stress of $1\frac{1}{4}$ tons per sq. in. is produced at the common surface which has a diameter of 6 in. If the compound tube is subjected to an internal pressure of 8500 lb. per sq. in., find the maximum hoop stress. (London Univ.)

For the outer cylinder,

$$p_x = 1\tfrac{1}{4} \text{ tons per sq. in. at } r = 3 \text{ in.,}$$

and

$$= 0 \text{ at } r = 4 \text{ in.}$$

Apply the formulæ

$$p_x = a + \frac{b}{r^2},$$

$$p_y = \frac{b}{r^2} - a.$$

$$\therefore\ p_x = 1{\cdot}25 = a + b/3^2,$$

and

$$0 = a + b/4^2.$$

By subtraction, $1{\cdot}25 = \frac{1}{9}b - \frac{1}{16}b.$

$$\therefore\ b = \frac{1{\cdot}25 \times 9 \times 16}{7} = 25{\cdot}7,$$

and $$a = -\frac{25{\cdot}7}{4^2} = -1{\cdot}606.$$

Substituting in $p_y = \frac{b}{r^2} - a$, when $r = 3$ in. and 4 in.,

$$p_y = \frac{25{\cdot}7}{3^2} - (-1{\cdot}606)$$

$$= 2{\cdot}86 + 1{\cdot}606$$

$$= 4{\cdot}47 \text{ tons per sq. in. tension,}$$

and $$p_y = \frac{25{\cdot}7}{4^2} - (-1{\cdot}606)$$

$$= 3{\cdot}21 \text{ tons per sq. in. tension.}$$

For the inner cylinder,

$$p_x = 1\tfrac{1}{4} \text{ tons per sq. in. at } r = 3 \text{ in.,}$$

and $$= 0 \text{ at } r = 2 \text{ in.}$$

$$\therefore\ p_x = 1{\cdot}25 = a + b/3^2,$$

and $$0 = a + b/2^2.$$

By subtraction, $1{\cdot}25 = \frac{1}{9}b - \frac{1}{4}b.$

$$\therefore\ b = -\frac{1{\cdot}25 \times 9 \times 4}{5} = -9,$$

and $$a = 9/4 = 2{\cdot}25.$$

Substituting in $p_y = \frac{b}{r^2} - a$, when $r = 2$ in. and 3 in.,

$$p_y = \frac{-9}{2^2} - 2{\cdot}25$$

$$= -4{\cdot}5 \text{ tons per sq. in. (compression),}$$

and $$p_y = \frac{-9}{3^2} - 2{\cdot}25$$

$$= -3{\cdot}25 \text{ tons per sq. in. (compression).}$$

The negative sign indicates compression, since the formulæ $p_x = a + b/r^2$ and $p_y = b/r^2 - a$ were deduced for an internal pressure, whilst the pressure on the inner cylinder due to the outer cylinder is external.

For the internal pressure of 8500 lb. per sq. in., neglecting the shrinkage stress,

$$p_x = 8500 \text{ lb. per sq. in. at } r = 2 \text{ in.,}$$

and $$= 0 \text{ at } r = 4 \text{ in.}$$

Then $$8500 = \frac{b}{2^2} + a,$$

and $$0 = \frac{b}{4^2} + a.$$

By subtraction, $$8500 = \tfrac{1}{4}b - \tfrac{1}{16}b.$$

$$\therefore\ b = \frac{8500 \times 16}{3} = 45{,}300,$$

and $$a = -\frac{45{,}300}{16} = -2830.$$

Substituting in $p_y = b/r^2 - a$ when $r = 2$ in., 3 in. and 4 in.,

$$p_y = \frac{45{,}300}{2^2} - (-2830)$$

$$= 11{,}330 + 2830$$

$$= 14{,}160 \text{ lb. per sq. in. tension}$$

$$\equiv 6{\cdot}32 \text{ tons per sq. in. tension;}$$

$$p_y = \frac{45{,}300}{3^2} - (-2830)$$

$$= 5030 + 2830$$

$$= 7860 \text{ lb. per sq. in. tension}$$

$$\equiv 3{\cdot}51 \text{ tons per sq. in. tension;}$$

$$p_y = \frac{45{,}300}{4^2} - (-2830)$$

$$= 2830 + 2830$$

$$= 5660 \text{ lb. per sq. in. tension}$$

$$\equiv 2{\cdot}53 \text{ tons per sq. in. tension.}$$

The net stress will be the algebraic sum of the stresses caused by shrinkage and by the internal pressure.

For the inner tube, at $r = 2$ in.,

$$\text{stress} = -4{\cdot}5 + 6{\cdot}32$$

$$= 1{\cdot}82 \text{ tons per sq. in. tension;}$$

at $r = 3$ in., $$\text{stress} = -3{\cdot}25 + 3{\cdot}51$$

$$= 0{\cdot}26 \text{ ton per sq. in. tension.}$$

For the outer tube, at $r = 3$ in.,

$$\text{stress} = 4{\cdot}47 + 3{\cdot}51$$
$$= 7{\cdot}98 \text{ tons per sq. in. tension;}$$

at $r = 4$ in.,
$$\text{stress} = 3{\cdot}21 + 2{\cdot}53$$
$$= 5{\cdot}74 \text{ tons per sq. in. tension.}$$

Maximum hoop stress is 7·98 tons per sq. in. tension at 3 in. radius in the outer tube.

4. Shrunk or Force Fits.

So that there is a radial pressure between the two parts forced together there must be a difference between the two diameters. The internal diameter of the outer tube must be smaller than the external diameter of the inner tube. The calculations to find the necessary difference in diameters are simpler when tubes are forced or shrunk on one another than when a tube is forced on to a solid shaft.

(*a*) *Tubes.*

Assuming no internal pressure, there will be only two stresses p_x and p_y acting on the material when tubes are forced on to one another. The stress p_y will produce a circumferential strain equal to p_y/E, and p_x will produce a strain of $p_x/(mE)$ in the circumference. The net circumferential strain will be

$$\frac{p_y}{E} + \frac{p_x}{mE}.$$

The change in the radius will be equal to

$$\left(\frac{p_y}{E} + \frac{p_x}{mE}\right) \times \text{(radius)}.$$

Example 5.—Find the difference in diameter to be allowed for shrinkage when a compound cylinder, 8 in. external diameter, 4 in. internal diameter, and 6 in. diameter at the junction of the two tubes, has a radial pressure of 2 tons per sq. in. at the junction. $E = 13{,}000$ tons per sq. in., Poisson's ratio $1/m = \frac{1}{4}$.

For the outer cylinder,

$$p_x = 2 \text{ tons per sq. in. at } r = 3 \text{ in.,}$$

and
$$= 0 \text{ at } r = 4 \text{ in.}$$

Applying the formulæ
$$p_x = a + \frac{b}{r^2},$$

$$p_y = \frac{b}{r^2} - a,$$

$$p_x = 2 = a + b/3^2,$$

and $$0 = a + b/4^2.$$

By subtraction, $$2 = \tfrac{1}{9}b - \tfrac{1}{16}b.$$

$$\therefore\ b = \frac{2 \times 9 \times 16}{7} = 41{\cdot}1,$$

and $$a = -\frac{41{\cdot}1}{4^2} = -2{\cdot}57.$$

Substituting in $$p_y = \frac{b}{r^2} - a, \text{ when } r = 3 \text{ in. and } 4 \text{ in.,}$$

$$p_y = \frac{41{\cdot}1}{3^2} - (-2{\cdot}57)$$

$$= 4{\cdot}57 + 2{\cdot}57$$

$$= 7{\cdot}14 \text{ tons per sq. in. tension,}$$

and $$p_y = \frac{41{\cdot}1}{4^2} - (-2{\cdot}57)$$

$$= 5{\cdot}14 \text{ tons per sq. in. tension.}$$

For the inner cylinder,

$$p_x = 2 \text{ tons per sq. in. at } r = 3 \text{ in.,}$$

and $$= 0 \text{ at } r = 2 \text{ in.}$$

$$\therefore\ p_x = 2 = a + b/3^2,$$

and $$0 = a + b/2^2.$$

By subtraction, $$2 = \tfrac{1}{9}b - \tfrac{1}{4}b,$$

$$\therefore\ b = -\frac{2 \times 4 \times 9}{5} = -14{\cdot}4,$$

and $$a = \tfrac{1}{4} \times 14{\cdot}4 = 3{\cdot}6.$$

Substituting in $$p_y = \frac{b}{r^2} - a, \text{ when } r = 2 \text{ in. and } 3 \text{ in.,}$$

$$p_y = \frac{-14{\cdot}4}{2^2} - 3{\cdot}6$$

$$= -3{\cdot}6 - 3{\cdot}6$$

$$= -7{\cdot}2 \text{ tons per sq. in. compression,}$$

and $$p_y = \frac{-14{\cdot}4}{3^2} - 3{\cdot}6$$

$$= -5{\cdot}2 \text{ tons per sq. in. compression.}$$

Increase in 3 in. radius of outer cylinder due to stress

$$= \left(\frac{p_y}{E} + \frac{p_x}{mE}\right)3$$

$$= (7\cdot14 + \tfrac{2}{4})\frac{3}{13{,}000}$$

$$= \frac{7\cdot64 \times 3}{13{,}000}$$

$$= 1\cdot76 \times 10^{-3} \text{ in.}$$

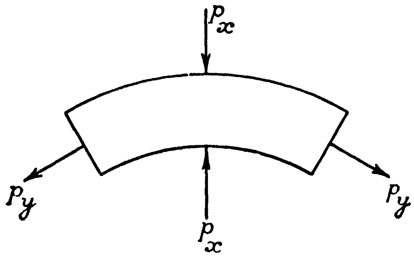


FIG. 9.5

Decrease in 3 in. radius of inner cylinder due to stress

$$= \left(\frac{p_y}{E} + \frac{p_x}{mE}\right)3$$

$$= (5\cdot2 - \tfrac{2}{4})\frac{3}{13{,}000}$$

$$= \frac{4\cdot7 \times 3}{13{,}000}$$

$$= 1\cdot085 \times 10^{-3} \text{ in.}$$

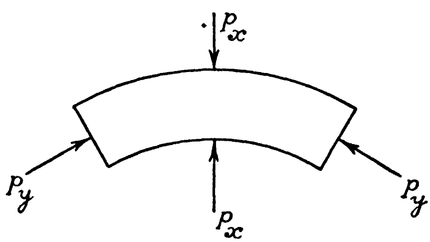


FIG. 9.6

Net difference in diameters $= 2(1\cdot76 + 1\cdot085)10^{-3}$

$$= 2 \times 2\cdot845 \times 10^{-3}$$

$$= 5\cdot69 \times 10^{-3} \text{ in.}$$

The difference in diameters is independent of Poisson's ratio, since $2/4$ is p_x/m and cancels out of the equation.

(*b*) *Solid shaft.*

This problem is greatly simplified if the compressibility of the shaft is neglected.

Example 6.—The external diameter of a steel hub is 8 in., and the pressure between the hub and solid shaft after shrinking is 2 tons per sq. in. If the diameter at the junction is 5 in., find the initial difference in diameters. $E = 13{,}000$ tons per sq. in. and $1/m = \frac{1}{4}$.

For the hub, $p_x = 2$ tons per sq. in. when $r = 2\frac{1}{2}$ in.,

and $= 0$ when $r = 4$ in.

Apply formulæ
$$p_x = a + b/r^2,$$
$$p_y = b/r^2 - a.$$
$$\therefore p_x = 2 = a + b/(2\tfrac{1}{2})^2,$$

and
$$0 = a + b/4^2.$$

By subtraction,
$$2 = b/(2\tfrac{1}{2})^2 - b/4^2.$$
$$\therefore b = \frac{2 \times 25 \times 16}{39} = 20{\cdot}5,$$

and
$$a = -\frac{20{\cdot}5}{16} = -1{\cdot}28.$$

Substituting in
$$p_y = b/r^2 - a, \text{ when } r = 2\tfrac{1}{2} \text{ in.},$$
$$p_y = \frac{20{\cdot}5}{(2\frac{1}{2})^2} - (-1{\cdot}28)$$
$$= 3{\cdot}28 + 1{\cdot}28$$
$$= 4{\cdot}56 \text{ tons per sq. in.}$$

Increase in $2\frac{1}{2}$ in. radius of hub due to stress
$$= \left(\frac{p_y}{E} + \frac{p_x}{mE}\right) 2\tfrac{1}{2}$$
$$= (4{\cdot}56 + \tfrac{2}{4}) \frac{2{\cdot}5}{13{,}000}$$
$$= \frac{5{\cdot}06 \times 2{\cdot}5}{13{,}000}$$
$$= 0{\cdot}975 \times 10^{-3} \text{ in.}$$

If the shaft is considered incompressible, the difference in diameters is $2 \times 0{\cdot}975 \times 10^{-3}$ in. $= 1{\cdot}95 \times 10^{-3}$ in.

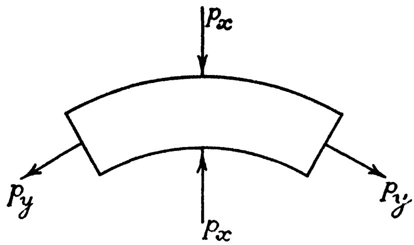


Fig. 9.7

Taking the same example, but allowing for the compressibility of the shaft material, it is necessary to obtain the value of the hoop or circumferential stress in the shaft material at the $2\frac{1}{2}$ in. radius.

Apply the formulæ

$$p_x = a + b/r^2,$$

$$p_y = b/r^2 - a,$$

where $p_r = 2$ tons per sq. in. at $r = 2\frac{1}{2}$ in.

When $r = 0$, the equation $p_x = a + b/r^2$ would give p_x equal to infinity, consequently b must also equal zero. Hence

$$p_x = a = -p_y.$$

The shaft material is therefore subjected to a constant radial and circumferential stress.

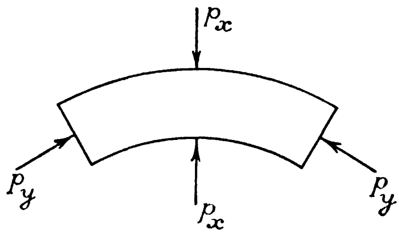


Fig. 9.8

$$\text{Decrease in radius at } 2\tfrac{1}{2} \text{ in.} = \left(\frac{p_y}{E_s} + \frac{p_x}{m_s E_s}\right) 2\tfrac{1}{2}$$

$$= \left(2 - \tfrac{2}{4}\right) \frac{2{\cdot}5}{13{,}000}$$

$$= 0{\cdot}289 \times 10^{-3} \text{ in.,}$$

if $1/m_s = \frac{1}{4}$ and $E_s = 13{,}000$ tons per sq. in.

$$\text{Total difference in radii} = 0{\cdot}975 \times 10^{-3} + 0{\cdot}289 \times 10^{-3}$$

$$= 1{\cdot}264 \times 10^{-3} \text{ in.}$$

$$\therefore \text{ Difference in diameter} = 2{\cdot}528 \times 10^{-3} \text{ in.}$$

Consequently there is an error of $0{\cdot}578 \times 10^{-3}$ in. or 22·9 per cent by the incompressibility method.

Example 7.—The external diameter of a steel hub is 8 in., and the internal diameter increases 0·005 in. when shrunk on to a solid steel shaft of 5 in. diameter. Find the reduction in diameter of the shaft, the radial pressure between the hub and shaft, and the hoop stress at the inner surface of the hub. $1/m = 0{\cdot}304$; $E = 13{,}000$ tons per sq. in. (London Univ.)

The increase in the internal radius of the hub

$$= {\cdot}0025 = \left(\frac{p_y}{E} + \frac{p_x}{mE}\right)2\tfrac{1}{2}$$

$$= (p_y + {\cdot}304p_x)\,\frac{2{\cdot}5}{13{,}000},$$

or
$$p_y + {\cdot}304p_x = 13. \qquad \text{(1)}$$

Apply the formulæ
$$p_x = a + b/r^2, \qquad \text{(2)}$$
$$p_y = b/r^2 - a, \qquad \text{(3)}$$

where $p_x = 0$ when $r = 4$ in. Then

$$0 = a + \tfrac{1}{16}b,$$

or
$$a = -\tfrac{1}{16}b. \qquad \text{(4)}$$

From (1),
$$p_y = 13 - {\cdot}304p_x.$$

Substituting for p_y in (3),

$$13 - {\cdot}304p_x = b/r^2 - a.$$

$$\therefore\ -{\cdot}304p_x = b/r^2 - a - 13,$$

or
$$p_x = \frac{1}{{\cdot}304}(13 + a - b/r^2).$$

Substituting in (2),
$$a + b/r^2 = \frac{1}{{\cdot}304}(13 + a - b/r^2),$$

and
$$a = -\tfrac{1}{16}b \qquad \text{[from (4)]}.$$

$$\therefore\ -\frac{b}{16} + \frac{b}{r^2} = \frac{1}{{\cdot}304}\left(13 - \frac{b}{16} - \frac{b}{r^2}\right),$$

or
$${\cdot}304b\left(\frac{1}{r^2} - \frac{1}{16}\right) = 13 - b\left(\frac{1}{r^2} + \frac{1}{16}\right).$$

$$\therefore\ \frac{1{\cdot}304b}{r^2} + \frac{{\cdot}696b}{16} = 13.$$

$$\therefore\ b = 13\left(\frac{1}{\dfrac{1{\cdot}304}{r^2} + \dfrac{{\cdot}696}{16}}\right),$$

$$a = -\tfrac{1}{16}b.$$

The radial and hoop stresses are required for the $2\frac{1}{2}$ in. radius. Substituting in the equations for a and b the value $r = 2\frac{1}{2}$ in.,

$$b = 13\left(\frac{1}{\frac{1 \cdot 304}{(2\frac{1}{2})^2} + \frac{\cdot 696}{16}}\right)$$

$$= 13\left(\frac{1}{\cdot 209 + \cdot 0435}\right)$$

$$= 51 \cdot 5,$$

$$a = -\tfrac{1}{16} \times 51 \cdot 5$$

$$= -3 \cdot 22.$$

Therefore from (2),
$$p_x = -3 \cdot 22 + 51 \cdot 5/(2 \cdot 5)^2$$
$$= -3 \cdot 22 + 8 \cdot 25$$
$$= 5 \text{ tons per sq. in.}$$

For the hub,
$$p_y = b/r^2 - a$$
$$= \frac{51 \cdot 5}{(2 \cdot 5)^2} - (-3 \cdot 22)$$
$$= 8 \cdot 25 + 3 \cdot 22$$
$$= 11 \cdot 5 \text{ tons per sq. in. tension.}$$

For the shaft,
$$p_x = a + b/r^2,$$
$$p_y = b/r^2 - a,$$

where $p_x = 5$ tons per sq. in. at $r = 2\frac{1}{2}$ in., and $b = 0$. Hence

$$p_x = a = -p_y.$$

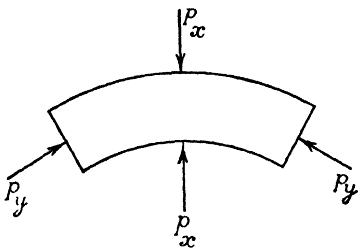


Fig. 9.9

$$\text{Decrease in radius} = \left(\frac{p_y}{E_s} - \frac{p_x}{m_s E_s}\right) 2\tfrac{1}{2}$$

$$= (5 - 5 \times \cdot 304)\,\frac{2 \cdot 5}{13{,}000}$$

$$= 0 \cdot 67 \times 10^{-3} \text{ in.}$$

$\therefore$ **Reduction in diameter of shaft** $= 1 \cdot 34 \times 10^{-3}$ **in.**

5. Pre-stressing.

The metal is generally considered to be sufficiently pre-stressed when a permanent increase in the diameter of 4 per cent is obtained.

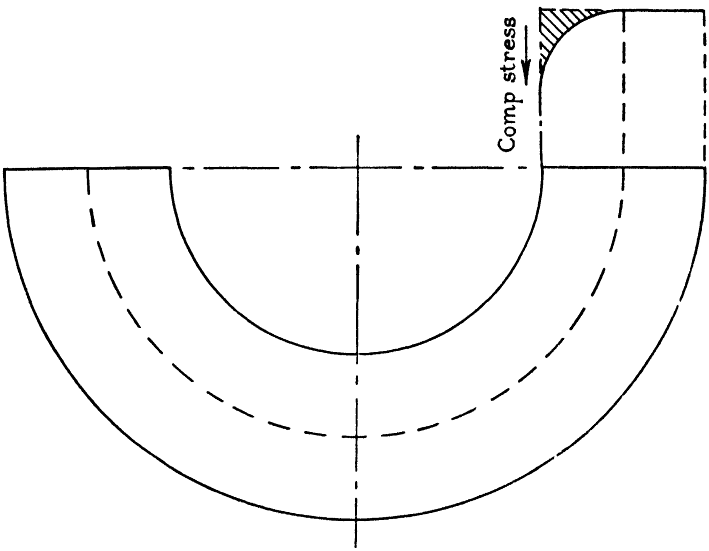


Fig. 9.10

The stress distribution before the application of an internal pressure will be as shown in fig. 9.10. When an internal pressure is applied, the stress distribution will approximate to that shown in fig. 9.11.

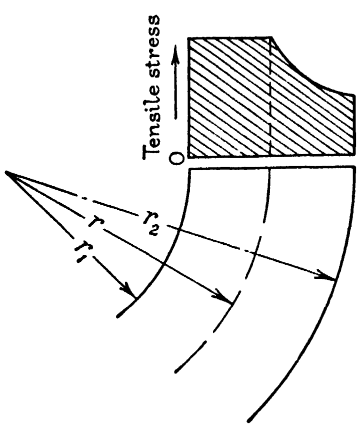


Fig. 9.11

6. Wire-wound Thick Cylinders.

The general equations for stresses in thick cylinders may be used, but it will be necessary to state the manner in which the tension in the wire varies. The simplest case occurs when the tension in the wire during winding is constant.

(*a*) *Tension constant.*

The wire will produce an external pressure on the cylinder, consequently compressive hoop stresses will be set up in the tube. If the combined wire and tube be considered as a thick cylinder, then the stresses are

$$p_x = \frac{p_1 r_2^2}{r_1^2 - r_2^2}\left[\frac{r_1^2 - r^2}{r^2}\right]$$

and

$$p_y = -\frac{p_1 r_2^2}{r_1^2 - r_2^2}\left[\frac{r^2 + r_1^2}{r^2}\right]$$

(see thick-cylinder equations, p. 194, with p_1 in place of p_2)

$$= -p_x\left(\frac{r^2 + r_1^2}{r_1^2 - r^2}\right)$$

$$= p_x\left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right).$$

If T be the initial tensile stress in the wire, the final tensile hoop stress in the winding at radius r is less than T by the hoop compressive stress set up by an external pressure

$$= T - p_x\left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right).$$

But

$$p_y = p_x + r\frac{dp_x}{dr}$$

for an external pressure (*Materials and Structures*, Chap. VIII). Hence

$$-\left(p_x + r\frac{dp_x}{dr}\right) = T - p_x\left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right).$$

The left-hand side is negative because p_y for an external pressure is compressive and T is tensile.

$$\therefore\ -r\frac{dp_x}{dr} = T - p_x\left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right) + p_x$$

$$= T - \frac{p_x \cdot 2r_1^2}{r^2 - r_1^2}.$$

$$\therefore\ -\frac{dp_x}{dr} + \frac{2r_1^2}{r(r^2 - r_1^2)}\cdot p_x = \frac{T}{r}.$$

Multiplying throughout by $-\dfrac{r^2}{r^2 - r_1^2}$,

$$\frac{r^2}{r^2 - r_1^2} \cdot \frac{dp_x}{dr} - \frac{2r_1^2 r^2}{r(r^2 - r_1^2)^2} \cdot p_x = -\frac{Tr^2}{(r^2 - r_1^2)r}.$$

$$\therefore \frac{d}{dr}\left(\frac{r^2}{r^2 - r_1^2} \cdot p_x\right) = \frac{-Tr}{r^2 - r_1^2}.$$

Hence $$\frac{r^2}{(r^2 - r_1^2)} \cdot p_x = -\tfrac{1}{2}T \log_e (r^2 - r_1^2) + A.$$

The radial pressure p_x equals zero when r equals the external radius (r_3) of the compound cylinders.

$$\therefore\ 0 = -\tfrac{1}{2}T \log_e (r_3^2 - r_1^2) + A,$$

$$\therefore\ A = \tfrac{1}{2}T \log_e (r_3^2 - r_1^2).$$

Therefore the complete solution is

$$\frac{r^2}{r^2 - r_1^2} \cdot p_x = -\tfrac{1}{2}T \log_e (r^2 - r_1^2) + \tfrac{1}{2}T \log_e (r_3^2 - r_1^2)$$

$$= \tfrac{1}{2}T \log_e \left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right).$$

$$\therefore\ p_x = \frac{(r^2 - r_1^2)T}{2r^2} \log_e \left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right).$$

$$\text{Hoop stress} = T - p_x\left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right)$$

$$= T - \left\{\frac{r^2 - r_1^2}{2r^2} T \log_e \left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right)\right\} \left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right)$$

$$= T\left\{1 - \left(\frac{r^2 + r_1^2}{2r^2}\right) \log_e \left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right)\right\}.$$

These equations will hold for all radii between r_2 and r_3, but not between r_1 and r_2, since T will be zero at those radii, that is, in the tube.

The stresses in the tube due to the wire winding may be found by treating it as a thick cylinder with an external pressure on the tube. The pressure is given by the radial pressure at $r = r_2$.

If an internal pressure be generated inside the cylinder, then the compound cylinder is treated as a solid cylinder, and the radial and hoop stresses calculated. The final stresses are the algebraic sums of the stresses due to the wire and the internal pressure.

Example 8.—A steel tube, 3 in. external diameter and 2 in. internal diameter, is closely wound with steel wire 0·05 in. diameter. The external diameter of the winding is 4 in. Find the circumferential stresses in the tube or wire at 1 in., $1\frac{1}{2}$ in. and 2 in. radii if the tension at which the wire is wound is 3000 lb. per sq. in., and an internal pressure of 3000 lb. per sq. in. acts in the cylinder.

The circumferential stresses due to the internal pressure may be found from the equations

$$p_x = b/x^2 + a,$$

$$p_y = b/x^2 - a,$$

where $p_x = 3000$ when $x = 1$ in.,

and $p_x = 0$ when $x = 2$ in.

$$\therefore\ p_x = 3000 = b/1^2 + a,$$

and $0 = b/2^2 + a.$

By subtraction, $3000 = b - \frac{1}{4}b,$

$$\therefore\ b = 4000,$$

and $a = -1000.$

When $x = 1$ in., $1\frac{1}{2}$ in. and 2 in.,

$$p_y = \frac{4000}{1} + 1000 = 5000 \text{ lb. per sq. in. tension,}$$

$$p_y = \frac{4000}{(\frac{3}{2})^2} + 1000 = 2778 \text{ lb. per sq. in. tension,}$$

and $p_y = \frac{4000}{(2)^2} + 1000 = 2000$ lb. per sq. in. tension.

Circumferential stress in wire when radii are $1\frac{1}{2}$ in. and 2 in. is given by

$$T\left[1 - \frac{r^2 + r_1^2}{2r^2}\log_e\left(\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right)\right]$$

$$= 3000\left[1 - \frac{(\frac{3}{2})^2 + 1^2}{2 \times (\frac{3}{2})^2}\log_e\left(\frac{2^2 - 1^2}{(\frac{3}{2})^2 - 1^2}\right)\right]$$

$$= 3000[1 - \tfrac{13}{18}\log_e \tfrac{12}{5}]$$

$$= 3000[1 - 0{\cdot}631]$$

$$= 1107 \text{ lb. per sq. in. tension.}$$

Circumferential stress at 2 in. radius

$$= T\left[1 - \frac{2^2 + 1^2}{2 \times (2)^2}\log_e\left(\frac{2^2 - 1^2}{2^2 - 1^2}\right)\right]$$

$$= T[1 - \tfrac{5}{8}\log_e 1]$$

$$= T[1 - 0]$$

$$= 3000 \text{ lb. per sq. in. tension.}$$

Radial stress at $1\frac{1}{2}$ in. due to wire

$$= \frac{r^2 - r_1^2}{2r^2} T \log_e \left[\frac{r_3^2 - r_1^2}{r^2 - r_1^2}\right]$$

$$= \frac{(\frac{3}{2})^2 - 1^2}{2 \times (\frac{3}{2})^2} 3000 \log_e \left[\frac{2^2 - 1^2}{(\frac{3}{2})^2 - 1^2}\right]$$

$$= \tfrac{5}{18} \times 3000 \log_e \tfrac{12}{5}$$

$$= \tfrac{2500}{3} \log_e \tfrac{12}{5}$$

$$= 729{\cdot}5 \simeq 730 \text{ lb. per sq. in.}$$

The circumferential stress in tube at $1\frac{1}{2}$ in. and 2 in. radii due to wire winding is found from

$$p_x = b/x^2 + a,$$

$$p_y = a - b/x^2,$$

since the pressure induced by the wire is external, where

$$p_x = 730 \text{ lb. per sq. in. at } 1\tfrac{1}{2} \text{ in. radius,}$$

and $\quad p_x = 0$ at 1 in. radius.

$$\therefore\ 730 = \frac{b}{(\frac{3}{2})^2} + a,$$

$$0 = b/1^2 + a.$$

By subtraction, $\quad 730 = 4b/9 - b,$

$$\therefore\ b = -1314,$$

and $\quad a = 1314.$

$$p_y = 1314 - \frac{(-1314)}{(\frac{3}{2})^2}$$

$$= 1898 \text{ lb. per sq. in. compression at } 1\tfrac{1}{2} \text{ in. radius,}$$

and $\quad p_y = 1314 - \left(\frac{-1314}{1^2}\right)$

$$= 2628 \text{ lb. per sq. in. compression at 1 in. radius.}$$

Net stresses equal the algebraic sum of the stresses due to the pressure and the wire winding.

In the tube at 1 in. radius,

$$\text{stress} = 5000 - 2628$$

$$= 2372 \text{ lb. per sq. in. tension,}$$

at $1\frac{1}{2}$ in. radius, $\quad \text{stress} = 2778 - 1898$

$$= 880 \text{ lb. per sq. in. tension,}$$

In the wire at $1\frac{1}{2}$ in. radius,

$$\text{stress} = 2778 + 1107$$
$$= 3885 \text{ lb. per sq. in. tension,}$$

at 2 in. radius,
$$\text{stress} = 2000 + 3000$$
$$= 5000 \text{ lb. per sq. in. tension.}$$

(*b*) *Constant stress difference in winding.*

If the stress difference in the wire is to be constant throughout its length when an internal pressure is applied, then $p_y - p_x$ equals a constant.

The stresses in the winding when subjected to both the internal and winding pressures are connected by

$$p_y = p_x + r\frac{dp_x}{dr},$$

and
$$p_y - p_x = f = \text{a constant.}$$

$$\therefore r\frac{dp_x}{dr} = f,$$

i.e.
$$p_x = f\log_e r + A.$$

But $p_x = 0$ when $r = r_3$,

$$\therefore A = -f\log_e r_3.$$

$$\therefore p_x = f\log_e\frac{r}{r_3},$$

and
$$p_y = f\log_e\frac{r}{r_3} + f$$
$$= f\left(\log_e\frac{r}{r_3} + 1\right).$$

For the internal pressure,

$$p_{x_1} = \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left(\frac{r_3^2 - r^2}{r^2}\right),$$

$$p_{y_1} = \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left(\frac{r_3^2 + r^2}{r^2}\right).$$

Initial stresses p_{x_2} and p_{y_2} in the windings are

$$p_{x_2} = p_x - p_{x_1}$$
$$= f\log_e\frac{r}{r_3} - \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left[\frac{r_3^2 - r^2}{r^2}\right],$$

and $$p_{v_2} = p_v - p_{v_1}$$

$$= f\left[\log_e \frac{r}{r_3} + 1\right] - \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left[\frac{r_3^2 + r^2}{r^2}\right].$$

The initial tension in the wire at radius r will differ from p_{v_2} by the circumferential stress in the wire caused by the radial pressure produced by the windings between radius r and r_3.

The circumferential stress produced by an external pressure

$$= \frac{p_1 r_2^2}{r_2^2 - r_1^2}\left[\frac{r^2 + r_1^2}{r^2}\right] \text{ compression}$$

$$= \frac{r^2 + r_1^2}{r^2 - r_1^2} \cdot p_{x_2},$$

where p_{x_2} is the radial stress.

The initial tension in the wire at radius r

$$= p_{v_2} - \frac{r^2 + r_1^2}{r^2 - r_1^2} \cdot p_{x_2}$$

$$= f\left[\log_e \frac{r}{r_3} + 1\right] - \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left[\frac{r_3^2 + r^2}{r^2}\right] + \frac{r^2 + r_1^2}{r^2 - r_1^2}\left[f \log_e \frac{r}{r_3} - \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left(\frac{r_3^2 - r^2}{r^2}\right)\right]$$

$$= f\left[\log_e \frac{r}{r_3}\left(1 + \frac{r^2 + r_1^2}{r^2 - r_1^2}\right) + 1\right] - \frac{p_1 r_1^2}{r_3^2 - r_1^2}\left[\frac{r_3^2 + r^2}{r^2} + \left(\frac{r^2 + r_1^2}{r^2 - r_1^2}\right)\left(\frac{r_3^2 - r^2}{r^2}\right)\right]$$

$$= f\left[\frac{2r^2}{r^2 - r_1^2}\log_e \frac{r}{r_3} + 1\right] - \frac{2p_1 r_1^2}{r^2 - r_1^2}.$$

7. Wire-wound Thin Cylinders.

Thin cylinders may be reinforced in the same way as described for thick cylinders, but the calculations have to be modified because the stress across the wall face will be uniform.

(a) Cylinder wound with wire of same material.

Before the application of an internal pressure, the tension in the wire will cause a circumferential compressive stress in the cylinder walls.

Let f_c = circumferential compressive stress caused by wire winding.
l = cylinder length,
t = cylinder thickness,
d = internal diameter,
w = width of wire,
u = thickness of wire,
T = winding tension,
p = internal pressure,

then
$$f_c \times l \times t \times 2 = T \times \frac{l}{w} \times 2,$$
$$\therefore f_c = \frac{T}{ut}.$$

The initial tensile stress in the wire $= \dfrac{T}{wu}$.

The total bursting pressure, when the internal pressure is applied
$$= p \times d \times l.$$

The area of resistance will be the combined areas of the cylinder and wire, i.e.
$$2 \times l \times t + 2 \times \frac{l}{w} \times w \times u = 2l(t + u).$$

Hence tensile stress in wire and cylinder caused by internal pressure equals
$$\frac{pdl}{2l(t+u)} = \frac{pd}{2(t+u)}.$$

Now the wire is in tension and the cylinder is in compression due to the original wire winding, hence the final stresses in the wire and cylinder will be the algebraic sum of the stresses caused by the wire winding and the internal pressure.

$$\text{Tension in wire} = \frac{pd}{2(t+u)} + \frac{T}{uw},$$
$$\text{Tension in cylinder wall} = \frac{pd}{2(t+u)} + \frac{T}{wt}.$$

(*b*) *Cylinder wound with wire of different material.*

The calculations will be the same as for the first part when no internal pressure is acting. That is,

$$\text{compressive stress in walls of cylinder} = \frac{T}{wt}, \quad . \quad . \quad (1)$$

and
$$\text{tensile stress in wire} = \frac{T}{uw}. \quad . \quad (2)$$

As the materials are different from one another the strains will be different, and in consequence the differing values of the moduli of elasticity must be taken into consideration when dealing with the internal pressure. Assuming that the stresses are uniform over the cross-section of both the cylinder wall and the wire, then the strains will be proportional to the mean radii.

$$\text{The mean radius for cylinder} = \tfrac{1}{2}(d + t) = e_1. \quad . \quad . \quad (3)$$

$$\text{The mean radius for wire} = \tfrac{1}{2}d + t + \tfrac{1}{2}u = e_2. \quad . \quad (4)$$

Then

$$\frac{e_1}{e_2} = \frac{\dfrac{f_1}{E_1}}{\dfrac{f_2}{E_2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where f_1 and E_1 are the stress and modulus of elasticity for the cylinder, and f_2 and E_2 are the stress and modulus of elasticity for the wire. Hence

$$f_1 = \frac{e_1 E_1 f_2}{e_2 E_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The load taken by the cylinder and wire is pdl. Hence

$$f_1 \times \text{area of cylinder walls} + f_2 \times \text{area of wire} = pdl,$$

i.e.

$$f_1 \,.\, 2lt + f_2 \,.\, 2lu = pdl. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By substituting from equations (3), (4) and (6) in (7), a solution for the effect of the internal pressure may be found.

The net stress in both the cylinder walls and the wire will be the algebraic sum of the stress caused by the internal pressure and the stress caused by wire winding as given by equations (1) and (2).

Example 9.—A copper tube, 2·5 in. external diameter and 2·4 in. internal diameter, is closely wound with steel wire of 0·05 in. diameter. Find the initial tension in the wire if the maximum allowable stress in the copper is 1000 lb. per sq. in. and the internal pressure is 200 lb. per sq. in. E for steel $= 1{\cdot}5 \times E$ for copper.

The mean radius of the cylinder = 1·225 in.

The mean radius of the coil of wire = 1·275 in.

Assuming the strains are proportional to the mean radii,

$$\frac{1{\cdot}225}{1{\cdot}275} = \frac{f_1}{E_1} \bigg/ \frac{f_2}{E_2},$$

where f_1 = stress in cylinder wall,

E_1 = modulus of elasticity of copper,

f_2 = stress in wire,

E_2 = modulus of elasticity of steel

$= 1{\cdot}5E_1$.

$$\therefore \frac{1{\cdot}225}{1{\cdot}275} = \frac{f_1}{f_2} \times 1{\cdot}5.$$

$$\therefore f_1 = \frac{1{\cdot}225}{1{\cdot}9125} f_2.$$

Equating the total stress in the cylinder walls, plus the total stress in the wire, to the total bursting pressure,

$$f_1 \,.\, 2l \times {\cdot}05 + f_2 \,.\, 2l \times {\cdot}05 = p \times 2{\cdot}5 \times l,$$

$$\therefore 0{\cdot}1 f_1 + 0{\cdot}1 f_2 = 2{\cdot}5p.$$

Substituting for f_2,

$$0{\cdot}1 f_1 + \frac{1{\cdot}9125}{1{\cdot}225} f_1 \times 0{\cdot}1 = 2{\cdot}5p.$$

$$\therefore 0{\cdot}256 f_1 = 2{\cdot}5p$$

$$= 2{\cdot}5 \times 200.$$

$$\therefore f_1 = \frac{500}{0{\cdot}256}$$

$$= 1950 \text{ lb. per sq. in.}$$

Compressive stress in copper due to wire winding $= \dfrac{T}{wt}$,

where T = wire tension,

w = width of wire,

t = thickness of cylinder wall.

$$\therefore \text{Compressive stress} = \frac{T}{{\cdot}05 \times {\cdot}05}.$$

Net tensile stress in copper $= 1000 = 1950 - \dfrac{T}{{\cdot}0025}$.

$$\therefore T = 950 \times {\cdot}0025$$

$$= 2{\cdot}375 \text{ lb.}$$

EXAMPLES

1. Find the ratio of thickness to internal diameter of a thick tube subjected to an internal pressure when this pressure is $\frac{5}{8}$ of the maximum permissible circumferential stress. Find the increase in internal diameter of such a tube of 4 in. internal diameter when the internal pressure is 6 tons per sq. in. E = 13,400 tons per sq. in., Poisson's ratio = 0·286.
[0·55 : 1; 0·00322 in.]

2. Find the ratio of thickness to internal diameter for a tube subjected to internal pressure when the ratio of the internal pressure to the greatest circumferential stress is 0·5.

Find the alteration of thickness of metal in such a tube of 8 in. internal diameter when the internal pressure is 5 tons per sq. in. Poisson's ratio = 0·304, E = 13,000 tons per sq. in. (London Univ.)
[0·732 : 1; $3·12 \times 10^{-3}$ in.]

3. A thick-walled tube, 2 in. internal diameter and 3 in. external diameter, is to withstand an internal pressure of 1·5 tons per sq. in. Calculate (*a*) maximum circumferential stress; (*b*) minimum circumferential stress; (*c*) maximum shear stress.
[3·9 tons per sq. in; 2·4 tons per sq. in.; 2·7 tons per sq. in.]

4. A compound cylinder is to be made by shrinking an outer cylinder of 12 in. external diameter on to an inner tube of 6 in. internal diameter. Determine the common diameter at the junction if the greatest circumferential stress in the inner tube is to be two-thirds of the greatest circumferential stress in the outer tube. (London Univ.) [9 in.]

5. A steel cylinder, 8 in. external diameter and 6 in. internal diameter, has another cylinder 10 in. internal diameter shrunk on it. If the maximum tensile stress induced in the outer cylinder is 5 tons per sq. in., find the radial compressive stress between the cylinders.

Determine the circumferential stresses at inner and outer diameter of both cylinders, and show, by means of a diagram, how these stresses vary with the radius.

Calculate the necessary shrinkage allowance at the common surface. E = 13,000 tons per sq. in. (London Univ.)
[1·1 tons per sq. in. Inner cylinder: 5 tons per sq. in., 3·9 tons per sq. in. Outer cylinder: 5 tons per sq. in., 3·9 tons per sq. in. $5·63 \times 10^{-3}$ in.]

6. A long, straight cast-iron cylinder of 4 in. internal diameter and 6 in. external diameter is to be strengthened by shrinking on to it an outer steel cylinder so that it can withstand an internal pressure of 4 tons per sq. in. Calculate the least external diameter for the steel cylinder if the greatest stress difference is not to exceed 10 tons per sq. in. and the tensile stress in the cast iron is not to exceed 2 tons per sq. in.

7. A solid steel shaft of 1 in. diameter is pressed into a steel sleeve. Find the initial difference of diameters when the common radial pressure is 1500 lb. per sq. in. and the circumferential stress is 3000 lb. per sq. in. at the inside of the sleeve.

Find the axial compressive load that should be applied to the shaft to increase the radial pressure to 2000 lb. per sq. in. Poisson's ratio = 0·304. (London Univ.) [$3{\cdot}09 \times 10^{-4}$ in.; 4·25 tons per sq. in.]

8. A steel sleeve is pressed on to a solid steel shaft of 2 in. diameter. The radial pressure between the shaft and the sleeve is 2500 lb. per sq. in., and the hoop stress at the inner surface of the sleeve is 6000 lb. per sq. in. If an axial compressive load of 12,000 lb. is now applied to the shaft, determine the alteration of radial pressure. Poisson's ratio = 0·304. (London Univ.) [430 lb. per sq. in. increase.]

9. A copper tube of inside diameter 1·9 in. and outside diameter 2 in. is closely wound with steel wire of diameter 0·028 in. Find the tension at which the wire must be wound on the tube if an internal pressure of 200 lb. per sq. in. is required before the copper is subjected to a tensile stress, the tube being free to expand or contract axially. For steel, Young's modulus = 30×10^6 lb. per sq. in.; for copper, Young's modulus = 18×10^6 lb. per sq. in. (London Univ.) [2·44 lb.]

10. A copper tube, 1·5 in. external diameter and 1·4 in. internal diameter, is closely wound with steel wire, 0·03 in. diameter. Stating clearly the assumptions made, estimate the tension at which the wire must be wound if an internal gauge pressure of 300 lb. per sq. in. produces a tensile circumferential stress of 1000 lb. per sq. in. in the copper tube. E for steel = $1{\cdot}6 \times E$ for copper. (London Univ.) [2·1 lb.]

CHAPTER X

Struts

It has been shown in *Materials and Structures* ~~that~~ the derivation of formulæ to give the maximum load that struts will withstand are based on wide assumptions. Euler's formula assumes that the struts fail by bending without any reference to the direct stress, whilst other formulæ assume the proportion of direct stress and bending stress which causes failure. The equation for a pin-jointed strut was fully worked out mathematically, but for other end-fixings the final formulæ were obtained by a process of deduction. It will be better now to collect together these formulæ and produce them all in the mathematical manner.

1. Euler Formula.

(*a*) *Both ends pin-jointed.*

The bending moment at section XX is Py, hence

$$EI\frac{d^2y}{dx^2} = -Py,$$

or

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y$$

$$= -n^2y, \text{ where } n^2 = \frac{P}{EI}.$$

Then

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

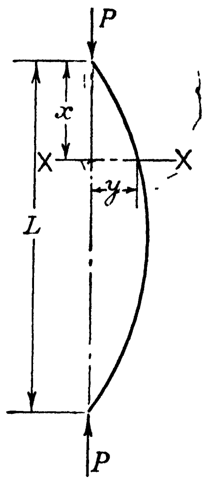


Fig. 10.1

The solution of this equation is $y = A\cos nx + B\sin nx$, where A and B are arbitrary constants.

The known conditions are $y = 0$ when $x = 0$ and $x = L$.

$$y = 0 = A\cos 0 + B\sin 0$$

$$= A + 0.$$

$$\therefore A = 0.$$

Also

$$y = 0 = B\sin nL.$$

B cannot be zero, $\therefore\ \sin nL = 0.$

$$\therefore\ nL = \pi \quad \text{(least value)},$$

and
$$n^2 = \frac{\pi^2}{L^2}.$$

But
$$n^2 = \frac{P}{EI},$$

$$\therefore\ P = \frac{\pi^2 EI}{L^2}.$$

(*b*) *One end fixed, the other free.*

The bending moment at XX is Py, so that

$$EI\frac{d^2y}{dx^2} = -Py,$$

or
$$EI\frac{d^2y}{dx^2} + Py = 0.$$

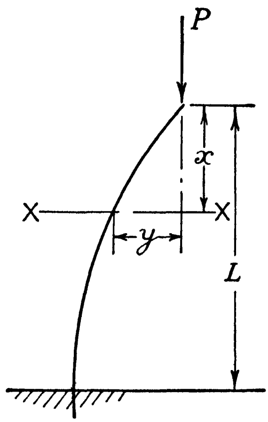


Fig. 10.2

This expression is the same as the one obtained in (*a*) and therefore will have the same solution, namely, $y = A\cos nx + B\sin nx$. But the known conditions are different; they are $y = 0$ when $x = 0$, and $dy/dx = 0$ when $x = L$.

When $y = 0$, $y = 0 = A + 0$,

$$\therefore\ A = 0.$$

Hence
$$y = B\sin nx,$$

and
$$\frac{dy}{dx} = nB\cos nx.$$

Substituting $x = L$ and $dy/dx = 0$,

$$\frac{dy}{dx} = 0 = nB\cos nL.$$

Now n and B are not zero,

$$\therefore\ \cos nL = 0,$$

or
$$nL = \tfrac{1}{2}\pi \quad \text{(least value)}.$$

$$\therefore\ n^2 = \frac{\pi^2}{4L^2}.$$

But
$$n^2 = \frac{P}{EI},$$

$$\therefore\ P = \frac{\pi^2 EI}{4L^2}.$$

(*c*) *Both ends fixed.*

In this case there will be fixing moments at each end, and by symmetry they will be equal, and equal to M, so that the bending moment at XX is $-Py + M$.

$$\therefore\ EI\frac{d^2y}{dx^2} = -Py + M,$$

or
$$\frac{d^2y}{dx^2} + n^2y - \frac{n^2M}{P} = 0, \text{ where } n^2 = \frac{P}{EI}.$$

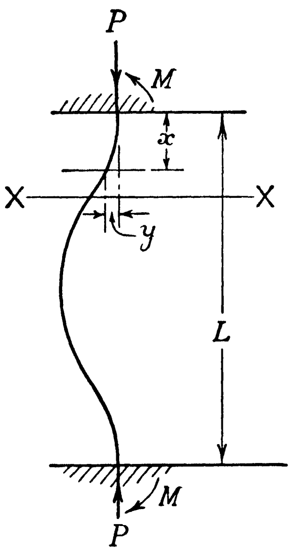


Fig. 10.3

The solution to this equation has two component parts: (i) complementary function, (ii) particular integral.

The complementary function is $y = A\cos nx + B\sin nx$, whilst the particular integral is $+M/P$. Hence the complete solution is

$$y = A\cos nx + B\sin nx + \frac{M}{P}.$$

It may be verified at once by actual differentiation that the above is the solution to this equation.

The known conditions of this problem are

$$y = 0 \text{ when } x = 0 \text{ and } L,$$

and
$$\frac{dy}{dx} = 0 \text{ when } x = 0 \text{ and } \tfrac{1}{2}L.$$

Substituting $y = 0$ and $x = 0$,

$$y = 0 = A + 0 + \frac{M}{P},$$

$$\therefore\ A = -\frac{M}{P}.$$

Hence
$$y = -\frac{M}{P}\cos nx + B\sin nx + \frac{M}{P},$$

and
$$\frac{dy}{dx} = n\frac{M}{P}\sin nx + nB\cos nx.$$

When $x = 0$, $\frac{dy}{dx} = 0$,

$$\therefore\ \frac{dy}{dx} = 0 = 0 + nB,$$

$$\therefore\ B = 0.$$

$$\therefore\ y = -\frac{M}{P}\cos nx + \frac{M}{P}.$$

When $x = L$, $y = 0 = -\frac{M}{P}\cos nL + \frac{M}{P} = \frac{M}{P}(1 - \cos nL)$.

But M and P are not zero,

$$\therefore\ 1 - \cos nL = 0.$$

$$\therefore\ \cos nL = 1,$$

i.e.
$$nL = 2\pi \quad \text{(least value)}.$$

$$\therefore\ n^2 = \frac{4\pi^2}{L^2}.$$

$$\therefore\ P = \frac{4\pi^2 EI}{L^2}.$$

(*d*) *One end fixed and one pin-jointed.*

Lateral movement is prevented at the pin-jointed end by the application of a force R.

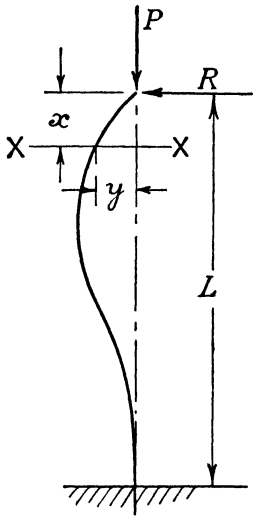


Fig 10.4

Therefore the bending moment at XX is $-Py + Rx$, so that

$$EI\frac{d^2y}{dx^2} = -Py + Rx,$$

or
$$\frac{d^2y}{dx^2} + n^2y - \frac{Rn^2x}{P} = 0, \text{ where } n^2 = \frac{P}{EI}.$$

The solution to this equation is

$$y = A\cos nx + B\sin nx + \frac{Rx}{P},$$

and the known conditions are

$$y = 0 \text{ when } x = 0 \text{ and } L,$$

and
$$\frac{dy}{dx} = 0 \text{ when } x = L.$$

$$\therefore\ y = 0 = A + 0 + 0,\ \therefore\ A = 0,$$

and
$$y = 0 = B\sin nL + \frac{RL}{P} \text{ when } x = L.$$

$$\therefore\ B \sin nL = -\frac{RL}{P}. \quad \ldots\ldots \quad (1)$$

$$\frac{dy}{dx} = nB \cos nx + \frac{R}{P},$$

$$\therefore\ \frac{dy}{dx} = 0 = nB \cos nL + \frac{R}{P} \text{ when } x = L.$$

Hence
$$B \cos nL = -\frac{R}{nP}. \quad \ldots\ldots \quad (2)$$

Dividing equation (1) by (2),

$$\tan nL = nL.$$

The value of nL must lie between 0 and $\frac{1}{2}\pi$, or π and $3\pi/2$, but, as the only value for which $\tan nL = nL$ between 0 and $\frac{1}{2}\pi$ is zero, the desired result must lie between π and $3\pi/2$.

Let
$$nL = \pi + \phi,$$
then
$$\tan(\pi + \phi) = \pi + \phi,$$
i.e.
$$\tan \phi = \pi + \phi.$$

To solve this equation it is advisable to draw graphs of $\tan \phi$ and $\pi + \phi$, and note where they cross.

When $\phi = 77° 27'$,

$$\tan \phi = 4{\cdot}4922,$$

and
$$\phi = 1{\cdot}3517 \text{ radians}.$$

$$\therefore\ \pi + \phi = 4{\cdot}4933 \text{ radians} \simeq \tan \phi.$$

Hence
$$nL = \pi + \phi$$
$$= 4{\cdot}4933.$$

$$\therefore\ n^2L^2 = (4{\cdot}4933)^2$$
$$= 20{\cdot}19$$
$$= 2 \times \pi^2 \text{ approximately.}$$

$$\therefore\ \frac{P}{EI} = \frac{2\pi^2}{L^2},$$

$$\therefore\ P = \frac{2\pi^2 EI}{L^2}.$$

2. Pin-jointed Strut with Eccentric Load.

Euler's formula is deduced on the assumption that the strut is axially loaded, but, if the load is not axial, then the problem may be solved in the following manner.

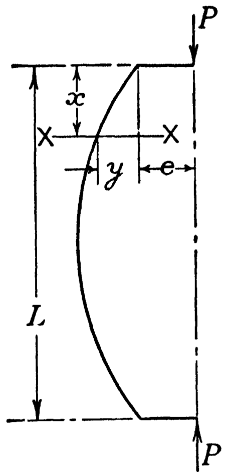


Fig. 10.5

Bending moment at XX is $-(Py + Pe)$.

$$\therefore\ EI\frac{d^2y}{dx^2} = -(Py + Pe),$$

or
$$\frac{d^2y}{dx^2} + n^2y + n^2e = 0, \text{ where } n^2 = \frac{P}{EI}.$$

The solution of this equation is

$$y = A\cos nx + B\sin nx - e.$$

Using the known conditions $y = 0$ when $x = 0$,

and
$$\frac{dy}{dx} = 0 \text{ when } x = \tfrac{1}{2}L,$$

$$y = 0 = A + 0 - e,$$

$$\therefore\ A = e.$$

$$\frac{dy}{dx} = -en\sin nx + nB\cos nx.$$

$$\therefore \frac{dy}{dx} = 0 = -ne \sin \tfrac{1}{2}nL + nB \cos \tfrac{1}{2}nL.$$

$$\therefore B = e \tan \tfrac{1}{2}nL.$$

$$\therefore y = e \cos nx + e \tan \tfrac{1}{2}nL \sin nx - e.$$

The maximum deflection will occur when $x = \frac{1}{2}L$, i.e.

$$y_{max} = e \cos \tfrac{1}{2}nL + e \tan \tfrac{1}{2}nL \sin \tfrac{1}{2}nL - e$$

$$= e[\sec \tfrac{1}{2}nL - 1].$$

The maximum bending moment $\quad -P\{e(\sec \frac{1}{2}nL - 1) + e\}$

$$= -Pe \sec \tfrac{1}{2}nL.$$

Maximum compressive bending stress $= Pe \sec \frac{1}{2}nL \times \dfrac{m_c}{I}$,

where m_c is the distance of the farthest material on the compression side from the neutral surface, and I is the second moment of area about the neutral surface.

Total compressive stress = bending stress + direct stress,

i.e. $$f_c = \frac{m_c}{I} Pe \sec \tfrac{1}{2}nL + \frac{P}{A},$$

where A is the area of the section.

$$\therefore Af_c = \frac{m_c}{k^2} Pe \sec \tfrac{1}{2}nL + P.$$

$$\therefore P = \frac{Af_c}{1 + \dfrac{m_c}{k^2} \cdot e \sec \tfrac{1}{2}nL}.$$

But $$n^2 = \frac{P}{EI},$$

$$\therefore P = \frac{Af_c}{1 + \dfrac{m_c}{k^2} e \sec \left(\tfrac{1}{2}L\sqrt{\dfrac{P}{EI}}\right)}.$$

It will be noticed that P occurs on both sides of the equation, hence it is difficult to obtain the value of P unless trial-and-error, or graphical, methods are used.

Example 1.—A strut 10 ft. long is constructed of steel tube, 3 in. outside diameter and $\frac{1}{8}$ in. thick. The ends are pin-jointed, but the end-load of 5 tons is applied eccentrically through a line parallel to and 0·1 in. away from the axis of the strut, which is initially straight. Find the deflection and the maximum stress at the centre of the length. E = 13,000 tons per sq. in.

$$y_{max} = e\left[\sec \tfrac{1}{2}nL - 1\right]$$
$$= e\left[\sec\left(\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right) - 1\right],$$

where
$$e = 0{\cdot}1 \text{ in.},$$
$$L = 120 \text{ in.},$$
$$P = 5 \text{ tons},$$
$$E = 13{,}000 \text{ tons per sq. in.},$$
$$I = \tfrac{1}{64}\pi[3^4 - (2\tfrac{3}{4})^4] = 1{\cdot}168 \text{ in.}^4$$
$$\therefore y_{max} = 0{\cdot}1\left[\sec\left(\tfrac{120}{2}\sqrt{\frac{5}{13{,}000 \times 1{\cdot}168}}\right) - 1\right]$$
$$= {\cdot}1[\sec 1{\cdot}09 - 1]$$
$$= {\cdot}1[2{\cdot}16 - 1]$$
$$= 0{\cdot}116 \text{ in.}$$

Again
$$P = \frac{Af_c}{1 + \frac{m_c}{k^2}e\sec\tfrac{1}{2}L\sqrt{\frac{P}{EI}}},$$

where
$$A = \tfrac{1}{4}\pi[3^2 - (2\tfrac{3}{4})^2] = 1{\cdot}129 \text{ sq. in.},$$
$$m_c = 1{\cdot}5 \text{ in.},$$

and
$$k^2 = \frac{1{\cdot}168}{1{\cdot}129} = 1{\cdot}035 \text{ in.}^2$$

Then
$$5 = \frac{1{\cdot}129 f_c}{1 + \frac{1{\cdot}5}{1{\cdot}035} \times 0{\cdot}1 \sec \frac{120}{2}\sqrt{\frac{5}{13{,}000 \times 1{\cdot}168}}},$$
$$\therefore 5 = \frac{1{\cdot}129 f_c}{1 + 0{\cdot}145 \times \sec 1{\cdot}09}$$
$$= \frac{1{\cdot}129 f_c}{1{\cdot}313}.$$
$$\therefore f_c = 5{\cdot}82 \text{ tons per sq. in.}$$

3. Various Strut Formulæ.

The empirical strut formulæ mentioned in *Materials and Structures* are the Rankine, Gordon and Johnson's parabolic formulæ, and comparative results for experimental work are shown in fig. 8 of Chap. IX of that book.

As struts fail by buckling, due to the load not being placed axially, or the strut not being initially straight, it is only natural that there should be a large number of formulæ which attempt to give a value for the buckling load. The better known of these, other than the ones mentioned above, are the Fidler, Robertson, Southwell, and Moncrieff formulæ.

Rankine's formula may be modified to cover eccentric loads, when the eccentricity is known, in the following manner:

$$P = \frac{Af_c}{1 + a\left(\frac{L}{k}\right)^2\left(1 + \frac{em_c}{k^2}\right)},$$

where e = eccentricity,

m_c = distance of farthest material from the neutral surface on the compressive side.

Fidler's formula is

$$P = \frac{A}{\mu}\{f - p_e - \sqrt{(f + p_e)^2 - 2\,\mu f p_e}\},$$

where f = maximum direct compressive stress of the material,

p_e = Euler's stress $= \frac{\pi^2 EI}{AL^2}$ for pin-jointed strut,

μ = constant = 1·2.

Southwell's formula is a modification of the formula developed for a strut with eccentric loading. It is an adaptation of the eccentric-loading formula for a tubular strut of outside diameter d, in which the stress f_c is considered to be the stress (p_y) at the limit of proportionality of the material; thus

$$P = \frac{Ap_y}{1 + \frac{d}{2k^2}e \sec \frac{1}{2}L\sqrt{\frac{P}{EI}}}.$$

The equivalent eccentricity for a pin-jointed strut is given by

$$\frac{\text{length}}{600} + \frac{\text{internal diameter}}{40}.$$

The *Robertson formula* assumes an initial curvature for the strut such that $y_0 = e_0 \cos(\pi x/L)$, where e_0 is the eccentricity at the centre.

Let the deflection at XX be y due to the load, so that the bending moment at XX

$$= P(y + y_0)$$

$$= P\left(y + e_0 \cos\frac{\pi x}{L}\right),$$

i.e. $$EI\frac{d^2y}{dx^2} = -P\left(y + e_0 \cos\frac{\pi x}{L}\right),$$

and $$\frac{d^2y}{dx^2} + n^2y + e_0n^2 \cos\frac{\pi x}{L} = 0,$$

where $n^2 = \dfrac{P}{EI}$.

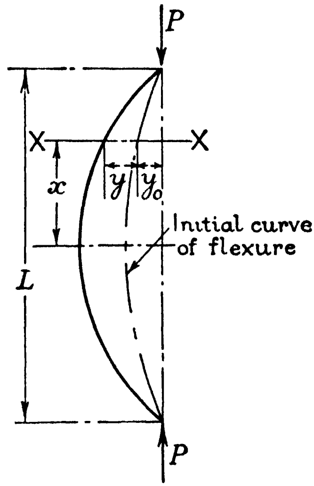


Fig. 10.6

The solution of this equation is

$$y = A\cos nx + B\sin nx + \frac{n^2e_0\cos\frac{\pi x}{L}}{\frac{\pi^2}{L^2} - n^2}.$$

Known conditions: $y = 0$ when $x = \frac{1}{2}L$,

and $$\frac{dy}{dx} = 0 \text{ when } x = 0.$$

$$\therefore \frac{dy}{dx} = -An\sin nx + Bn\cos nx - \frac{n^2e_0\frac{\pi}{L}\sin\frac{\pi x}{L}}{\frac{\pi^2}{L^2} - n^2},$$

and $$\frac{dy}{dx} = 0 \text{ when } x = 0, \ \therefore\ B = 0.$$

$$\therefore\ y = A\cos nx + \frac{n^2e_0\cos\frac{\pi}{L}x}{\frac{\pi^2}{L^2} - n^2}.$$

$$\therefore\ y = 0 = A\cos\tfrac{1}{2}nL + \frac{n^2e_0\cos\frac{\pi}{L}\cdot\frac{1}{2}L}{\frac{\pi^2}{L^2} - n^2}$$

$$= A\cos\tfrac{1}{2}nL + \frac{n^2e_0\cos\frac{1}{2}\pi}{\frac{\pi^2}{L^2} - n^2}$$

$$= A\cos\tfrac{1}{2}nL + 0.$$

Therefore A or $\cos \frac{1}{2}nL$ is zero. If $\cos \frac{1}{2}nL$ is zero, we get Euler's solution for a pin-jointed strut, but if A is zero, then

$$y = -\frac{n^2 e_0 \cos \frac{\pi}{L} x}{\frac{\pi^2}{L^2} - n^2}.$$

The total eccentricity at any point

$$= y + y_0$$

$$= \frac{n^2 e_0}{\frac{\pi^2}{L^2} - n^2} \cos \frac{\pi x}{L} + e_0 \cos \frac{\pi x}{L}$$

$$= \left[\frac{n^2}{\frac{\pi^2}{L^2} - n^2} + 1\right] e_0 \cos \frac{\pi x}{L}$$

$$= \left[\frac{\frac{P}{EI}}{\frac{\pi^2}{L^2} - \frac{P}{EI}} + 1\right] e_0 \cos \frac{\pi x}{L}$$

$$= \left[\frac{P}{\frac{\pi^2 EI}{L^2} - P} + 1\right] e_0 \cos \frac{\pi x}{L}$$

$$= \left[\frac{P}{P_e - P} + 1\right] e_0 \cos \frac{\pi x}{L},$$

where P_e = Euler's buckling load,

$$= \left[\frac{P_e}{P_e - P}\right] e_0 \cos \frac{\pi x}{L}.$$

The maximum bending moment will occur when $x = 0$, and is

$$\left(\frac{P_e}{P_e - P}\right) e_0 \, . \, P.$$

Maximum compressive bending stress

$$= \frac{P \, . \, P_e e_0}{P_e - P} \cdot \frac{m_c}{I},$$

where m_c = distance of farthest material from neutral surface on the compressive side.

$$\text{Direct compressive stress} = \frac{P}{A},$$

$$\therefore \text{ total compressive stress } p_1 = \frac{PP_e e_0}{P_e - P} \cdot \frac{m_c}{I} + \frac{P}{A}.$$

Let the ratio $\frac{e_0 m_c}{k^2} = \eta$, then

$$p_1 = \frac{PP_e}{P_e - P} \cdot \frac{\eta}{A} + \frac{P}{A}$$

$$= \frac{A^2 pp_e}{Ap_e - Ap} \cdot \frac{\eta}{A} + p$$

$$= \frac{pp_e}{p_e - p}\eta + p.$$

Let p_1 = compressive yield stress p_y, then

$$p_y = \frac{pp_e}{p_e - p}\eta + p,$$

or
$$p_y p_e - p_y p = \eta pp_e + pp_e - p^2.$$

$$\therefore p^2 - (1 + \eta)pp_e - p_y p + p_y p_e = 0,$$

i.e.
$$p^2 - p\{p_y + (1 + \eta)p_e\} + p_y p_e = 0,$$

$$\therefore p = \tfrac{1}{2}[p_y + (1 + \eta)p_e \pm \sqrt{\{p_y + (1 + \eta)p_e\}^2 - 4p_y p_e}].$$

For ductile materials which fail by compression,

$$p = \tfrac{1}{2}[p_y + (1 + \eta)p_e - \sqrt{\{p_y + (1 + \eta)p_e\}^2 - 4p_y p_e}],$$

where p = buckling stress,

p_y = compressive yield stress,*

p_e = Euler's buckling stress

$= \frac{\pi^2 EI}{AL^2}$ for pin-jointed strut

(the modification for end conditions is made in this value),

$\eta = \frac{0{\cdot}003L}{k}$, where L = length of strut,

k = least radius of gyration.

* Robertson has shown that for ductile materials, such as mild steel, it is the value of the yield stress in compression, and not the maximum stress, which determines failure. In the case of materials which do not have a real yield, 0·2 per cent proof stress, or the stress where the slope of the stress-strain graph is three times that for the elastic region.

Example 2.– Find the buckling load of a spruce strut 30 in. long and $1\frac{1}{2}$ in. square section. The compressive stress of the material is limited to 5000 lb. per sq. in., and the modulus of elasticity $1{\cdot}5 \times 10^6$ lb. per sq. in.

$$I = \tfrac{1}{12} \times (\tfrac{3}{2})^4 \text{ in.}^4,$$

$$A = (\tfrac{3}{2})^2 \text{ in.}^2,$$

$$k^2 = \frac{I}{A} = \tfrac{1}{12}(\tfrac{3}{2})^2 = \tfrac{3}{16} \text{ in.}^2,$$

$$\eta = \frac{0{\cdot}003L}{k} = \frac{0{\cdot}003 \times 30}{0{\cdot}433} = 0{\cdot}21,$$

$$p_e = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 \times 1{\cdot}5 \times 10^6 \times 3}{30^2 \times 16} = 3080 \text{ lb. per sq. in.}$$

Now
$$p = \tfrac{1}{2}\left[p_y + (1+\eta)p_e - \sqrt{\{p_y + (1+\eta)p_e\}^2 - 4p_y p_e}\right]$$
$$= \tfrac{1}{2}\left[5000 + (1 + {\cdot}21)3080 - \sqrt{\{5000 + (1 + {\cdot}21)3080\}^2 - 4 \times 5000 \times 3080}\right]$$
$$= \tfrac{1}{2}\left[5000 + 3720 - \sqrt{\{5000 + 3720\}^2 - 61{,}600{,}000}\right]$$
$$= \tfrac{1}{2}\left[8720 - \sqrt{76{,}056{,}000 - 61{,}600{,}000}\right]$$
$$= \tfrac{1}{2}(8720 - 3800)$$
$$= 2460 \text{ lb. per sq. in.}$$

$$\therefore \text{ The buckling load} = 2460 \times (\tfrac{3}{2})^2 = 5540 \text{ lb.}$$

Moncrieff formula.

An eccentricity of loading is assumed, but the curve of flexure is taken as a parabola. The basis of the formula is more or less rational, but an eccentricity factor from experimental results is used.

The equation to a parabola is $y = cx^2 + b$, where b and c are constants, and the axis of y is taken as the line of symmetry.
$$y = cx^2 + b.$$

Known conditions: $x = 0,\ y = \delta$;
$$x = \tfrac{1}{2}L,\ y = 0.$$

Hence $y = \delta = c\,.\,0 + b, \ \therefore\ b = \delta$,

and $y = 0 = c\,.\,\tfrac{1}{4}L^2 + \delta, \ \therefore\ c = -\dfrac{4\delta}{L^2}$.

Hence
$$y = -\frac{4\delta x^2}{L^2} + \delta.$$

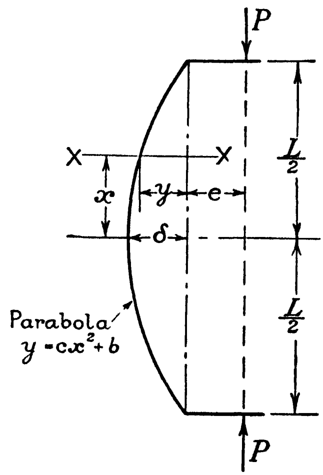


Fig. 10.7

With eccentricity e, the bending moment at XX is $P(y + e)$

$$= P\left(\delta - \frac{4\delta x^2}{L^2} + e\right).$$

Now $$EI\frac{d^2y}{dx^2} = \text{the bending moment}$$

$$= -P\left(\delta - \frac{4\delta x^2}{L^2} + e\right).$$

$$\therefore\ EI\frac{dy}{dx} = -P\left(\delta x - \frac{4\delta x^3}{3L^2} + ex\right) + A.$$

Now $\frac{dy}{dx} = 0$ when $x = 0$, $\quad \therefore\ A = 0$.

$$\therefore\ EIy = -P\left(\tfrac{1}{2}\delta x^2 - \frac{4\delta x^4}{12L^2} + \tfrac{1}{2}ex^2\right) + B.$$

Now $y = 0$ when $x = \frac{1}{2}L$.

$$\therefore\ EIy = 0 = -P\left(\tfrac{1}{2}\delta(\tfrac{1}{2}L)^2 - \frac{4\delta(\frac{1}{2}L)^4}{12L^2} + \tfrac{1}{2}e(\tfrac{1}{2}L)^2\right) + B.$$

$$\therefore\ B = P(\tfrac{1}{8}\delta L^2 - \tfrac{1}{48}\delta L^2 + \tfrac{1}{8}eL^2)$$

$$= P(\tfrac{5}{48}\delta L^2 + \tfrac{1}{8}eL^2).$$

$$\therefore\ EIy = -P\left(\tfrac{1}{2}\delta x^2 - \frac{\delta x^4}{3L^2} + \tfrac{1}{2}ex^2 - \tfrac{5}{48}\delta L^2 - \tfrac{1}{8}eL^2\right).$$

y is a maximum $(= \delta)$ when $x = 0$.

$$\therefore\ EI\delta = \tfrac{5}{48}P\delta L^2 + \tfrac{1}{8}PeL^2.$$

$$\therefore\ \delta = \frac{\frac{1}{8}PL^2e}{EI - \frac{5}{48}PL^2}.$$

Maximum bending moment $= P(\delta + e)$.

The maximum compressive stress will equal the sum of the bending and direct stresses, i.e.

$$f_c = \frac{P(\delta + e)m_c}{I} + \frac{P}{A},$$

where m_c is the distance of the farthest material on the compression side from the neutral surface.

As $\frac{P}{A} = p$, then

$$f_c - p = \frac{P}{I}(m_c\delta + m_c e)$$

$$= \frac{P}{I}\left\{\frac{\frac{1}{8}PL^2em_c}{EI - \frac{5}{48}PL^2} + m_c e\right\}$$

$$= \frac{P}{I}\left\{\frac{6PL^2em_c + m_c e(48EI - 5PL^2)}{48EI - 5PL^2}\right\}.$$

$$\therefore\ (f_c - p)(48EI - 5PL^2) = \frac{P}{I}\{m_ce(PL^2 + 48EI)\}.$$

$$\therefore\ (f_c - p)48EI - \frac{P}{I}\ m_ce\ 48EI = \frac{P}{I} m_cePL^2 + 5PL^2(f_c - p).$$

$$\therefore\ 48EAk^2\left(f_c - p - \frac{pm_ce}{k^2}\right) = P\left\{\frac{p}{k^2} m_ceL^2 + 5L^2(f_c - p)\right\}.$$

$$\therefore\ \frac{48EAk^2}{P}\left\{f_c - p - \frac{pm_ce}{k^2}\right\} = L^2\left\{\frac{pm_ce}{k^2} + 5(f_c - p)\right\},$$

or $\dfrac{L^2}{k^2} = \dfrac{\dfrac{48E}{p}\left(f_c - p - \dfrac{pm_ce}{k^2}\right)}{\dfrac{pm_ce}{k^2} + 5(f_c - p)}$. $\qquad \therefore\ \dfrac{L^2}{k^2} = \dfrac{48E\left\{\dfrac{f_c - p}{p} - \dfrac{m_ce}{k^2}\right\}}{5(f_c - p) + \dfrac{pm_ce}{k^2}}$.

Let the eccentricity factor $\dfrac{m_ce}{k^2} = 0{\cdot}6$, then

$$\frac{L^2}{k^2} = \frac{48E\left\{\dfrac{f_c - p}{p} - 0{\cdot}6\right\}}{5(f_c - p) + 0{\cdot}6p}$$

$$= \frac{48E(f_c - 1{\cdot}6p)}{p(5f_c - 4{\cdot}4p)}$$

$$= \frac{9{\cdot}6E(f_c - 1{\cdot}6p)}{p(f_c - {\cdot}88p)}.$$

$$\therefore\ \frac{L}{k} = \sqrt{\frac{9{\cdot}6E(f_c - 1{\cdot}6p)}{p(f_c - {\cdot}88p)}}.$$

Example 3.—A free ended strut of 3 in. diameter supports a load which induces a compressive stress of 20,000 lb. per sq. in. of cross-sectional area, the strut then being stressed to the buckling point. Find the ratio L/k when the material is mild steel with a maximum compressive stress of 60,000 lb. per sq. in. $E = 30 \times 10^6$ lb. per sq. in.

$$\frac{L}{k} = \sqrt{\frac{9{\cdot}6E(f_c - 1{\cdot}6p)}{p(f_c - {\cdot}88p)}}.$$

$f_c = 60{,}000$ lb. per sq. in.,

$p = 20{,}000$ lb. per sq. in.,

$$\therefore\ \frac{L}{k} = \sqrt{\left\{\frac{9{\cdot}6 \times 30 \times 10^6(60{,}000 - 1{\cdot}6 \times 20{,}000)}{20{,}000(60{,}000 - {\cdot}88 \times 20{,}000)}\right\}}$$

$$= \sqrt{\left\{\frac{9{\cdot}6 \times 30 \times 10^6 \times 28{,}000}{20{,}000 \times 42{,}400}\right\}} = \sqrt{9500}$$

$$= 97{\cdot}5.$$

Example 4.—Plot graphs of buckling stress against length for a free-ended steel strut made of steel tube of 1 in. outside diameter, ·036 in. thick. Use (*a*) Euler, (*b*) Rankine, (*c*) Southwell, (*d*) Robertson, and (*e*) Moncrieff formulæ. Consider the range 0 to 60 in. for the length.

Compressive yield stress – 18 tons per sq. in.;

maximum compressive stress 30 tons per sq. in.;

modulus of elasticity = 13,000 tons per sq. in.

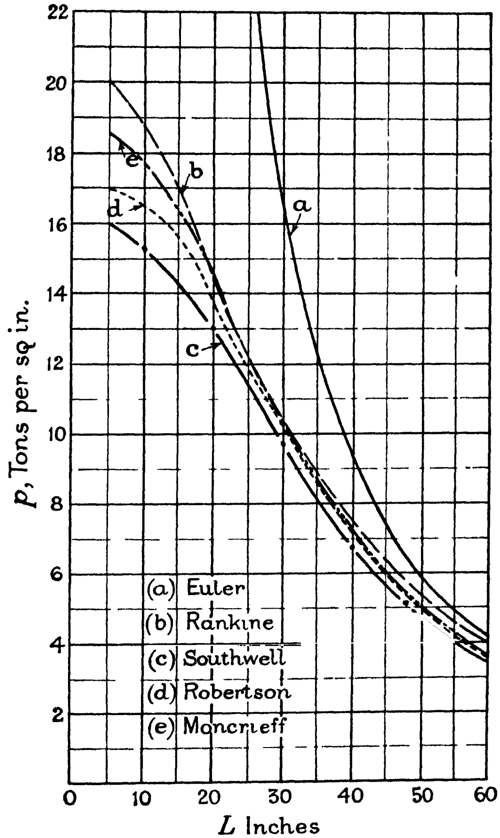


Fig. 10.8

(*a*) $p = \dfrac{\pi^2 E k^2}{L^2}$. $k^2 - \frac{1}{16}(1^2 + 0{\cdot}928^2) = 0{\cdot}116 \text{ in.}^2$

$\therefore\ p = p_e = \dfrac{14{,}900}{L^2}$.

(*b*) $$p = \frac{21}{1 + \frac{1}{7500}\left(\frac{L^2}{\cdot 116}\right)}$$ (where the value of f_c is taken as a constant of the Rankine formula equal to 21 tons per sq. in.)

$$= \frac{21}{1 + \frac{L^2}{870}}.$$

(*c*) $$p = \frac{p_y}{1 + \frac{d}{2k^2} e \sec \frac{1}{2}L\sqrt{\frac{p}{Ek^2}}} = \frac{18}{1 + \frac{1}{\cdot 232} e \sec \frac{1}{2}L\sqrt{\frac{p}{1510}}}.$$

$$e = \frac{L}{600} + \frac{\cdot 928}{40} = \frac{L}{600} + \cdot 0232.$$

(*d*) $$p = \tfrac{1}{2}\left[p_y + (1 + \eta)p_e - \sqrt{\{p_y + (1 + \eta)p_e\}^2 - 4p_y p_e}\right]$$

$$= \tfrac{1}{2}\left[18 + \left(1 + \frac{\cdot 003L}{\cdot 341}\right)\frac{14,900}{L^2} - \sqrt{\left\{18 + \left(1 + \frac{\cdot 003L}{\cdot 341}\right)\frac{14,900}{L^2}\right\}^2 - 4 \times 18 \times \frac{14,900}{L^2}}\right]$$

$$= \tfrac{1}{2}\left[18 + \frac{14,900}{L^2} + \frac{131}{L} - \sqrt{\left\{18 + \frac{14,900}{L^2} + \frac{131}{L}\right\}^2 - \frac{1,070,000}{L^2}}\right].$$

(*e*) $$\frac{L}{k} = \sqrt{\frac{9\cdot 6E(f_c - 1\cdot 6p)}{p(f_c - \cdot 88p)}},$$

$$\therefore \frac{L}{\cdot 341} = \sqrt{\frac{9\cdot 6 \times 13,000(30 - 1\cdot 6p)}{p(30 - \cdot 88p)}},$$

$$\therefore \frac{L^2}{\cdot 116} = \frac{3,740,000 - 200,000p}{30p - \cdot 88p^2},$$

$$\therefore L^2 = \frac{435,000 - 23,200p}{30p - \cdot 88p^2}.$$

4. Struts with Lateral Loads.

(*a*) *Concentrated load at the centre.*

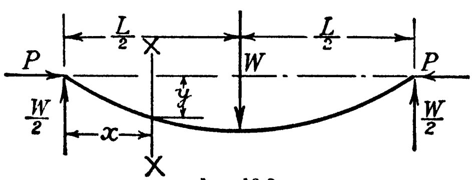


Fig. 10.9

For section XX, the bending moment is $Py + \frac{1}{2}Wx$, i.e.

$$EI\frac{d^2y}{dx^2} = -Py - \tfrac{1}{2}Wx.$$

$$\therefore \frac{d^2y}{dx^2} + n^2y + \frac{Wn^2x}{2P} = 0, \text{ where } n^2 = \frac{P}{EI}.$$

The solution of this equation is

$$y = A\cos nx + B\sin nx - \frac{Wx}{2P}.$$

The known conditions are $x = 0,\ y = 0$;

$$x = \tfrac{1}{2}L,\ \frac{dy}{dx} = 0.$$

Substituting for x and y, $A = 0$.

$$\therefore \frac{dy}{dx} = nB\cos nx - \frac{W}{2P}.$$

Substituting for x and $\frac{dy}{dx}$,

$$0 = nB\cos\tfrac{1}{2}nL - \frac{W}{2P},$$

$$\therefore B = \frac{W}{2Pn\cos\frac{1}{2}nL}.$$

$$\therefore y = \frac{W}{2Pn\cos\frac{1}{2}nL}\sin nx - \frac{Wx}{2P}.$$

The bending moment at XX

$$= P\left\{\frac{W\sin nx}{2Pn\cos\frac{1}{2}nL} - \frac{Wx}{2P}\right\} + \tfrac{1}{2}Wx.$$

The maximum bending moment will occur when $x = \frac{1}{2}L$, and equals

$$P\left\{\frac{W\sin\frac{1}{2}nL}{2Pn\cos\frac{1}{2}nL} - \frac{\frac{1}{2}WL}{2P}\right\} + \tfrac{1}{4}WL = \frac{W\tan\frac{1}{2}nL}{2n} - \tfrac{1}{4}WL + \tfrac{1}{4}WL$$

$$= \frac{W\tan\frac{1}{2}L\sqrt{\frac{P}{EI}}}{2\sqrt{\frac{P}{EI}}}$$

$$= \tfrac{1}{2}W\sqrt{\frac{EI}{P}}\cdot\tan\tfrac{1}{2}L\sqrt{\frac{P}{EI}}.$$

Hence maximum compressive bending stress

$$= \left(\tfrac{1}{2}W\sqrt{\frac{EI}{P}}\,.\tan\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right)\frac{m_c}{I},$$

where m_c is the distance of the farthest material on the compression side from the neutral surface.

$$\text{The direct compressive stress} = \frac{P}{A}.$$

Therefore total compressive stress

$$f_c = \left\{\tfrac{1}{2}W\sqrt{\frac{EI}{P}}\,.\tan\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right\}\frac{m_c}{I} + \frac{P}{A}.$$

(*b*) *Uniformly distributed load.*

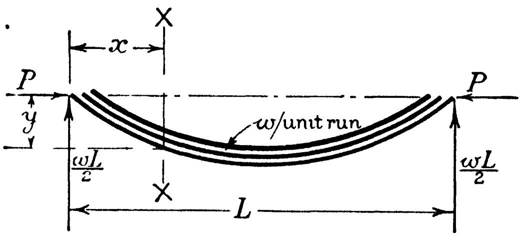


FIG. 10.10

For section XX, the bending moment is $Py + \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$.

$$\therefore\; EI\frac{d^2y}{dx^2} = -\{Py + \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2\}$$

and $$\frac{d^2y}{dx^2} + n^2y + \frac{n^2wLx}{2P} - \frac{n^2wx^2}{2P} = 0, \text{ where } n^2 = \frac{P}{EI}.$$

The solution of this equation is

$$y = A\cos nx + B\sin nx - \frac{wLx}{2P} + \frac{wx^2}{2P} - \frac{w}{n^2P}.$$

The known conditions are

$$y = 0 \text{ when } x = 0,$$

and $$\frac{dy}{dx} = 0 \text{ when } x = \tfrac{1}{2}L.$$

Substituting for y and x,

$$y = 0 = A - \frac{w}{n^2P},$$

$$\therefore A = \frac{w}{n^2P}.$$

$$\frac{dy}{dx} = -\frac{w}{n^2P} n \sin nx + nB \cos nx - \frac{wL}{2P} + \frac{wx}{P}.$$

Substituting for dy/dx and x,

$$\frac{dy}{dx} = 0 = -\frac{wn}{n^2P} \sin \tfrac{1}{2}nL + nB \cos \tfrac{1}{2}nL - \frac{wL}{2P} + \frac{wL}{2P}.$$

$$\therefore B = \frac{w}{n^2P} \tan \tfrac{1}{2}nL.$$

$$\therefore y = \frac{w}{n^2P} \cos nx + \frac{w}{n^2P} \tan \tfrac{1}{2}nL \sin nx - \frac{wLx}{2P} + \frac{wx^2}{2P} - \frac{w}{n^2P}.$$

The maximum deflection will occur when $x = \frac{1}{2}L$, and is

$$y_{\max} = \frac{w}{n^2P} \{\cos \tfrac{1}{2}nL + \tan \tfrac{1}{2}nL \sin \tfrac{1}{2}nL\} - \frac{wL^2}{4P} + \frac{wL^2}{8P} - \frac{w}{n^2P}$$

$$= \frac{w}{n^2P} \{\sec \tfrac{1}{2}nL - 1\} - \frac{wL^2}{8P}.$$

The maximum bending moment will occur at the centre and is

$$Py_{\max} + \tfrac{1}{2}wL \,.\, \tfrac{1}{2}L - \tfrac{1}{2}w(\tfrac{1}{2}L)^2 = P\left\{\frac{w}{n^2P}(\sec \tfrac{1}{2}nL - 1) - \frac{wL^2}{8P}\right\} + \tfrac{1}{8}wL^2$$

$$= \frac{w}{n^2}(\sec \tfrac{1}{2}nL - 1).$$

The maximum compressive stress equals the sum of the bending and direct stresses, i.e.

$$f_c = \frac{w}{n^2}(\sec \tfrac{1}{2}nL - 1)\frac{m_c}{I} + \frac{P}{A}.$$

$$= \frac{wEI}{P}\left(\sec \tfrac{1}{2}L\sqrt{\frac{P}{EI}} - 1\right)\frac{m_c}{I} + \frac{P}{A},$$

where m_c is the distance of the farthest material on the compression side from the neutral surface, and A is the area of the section.

Example 5.—A horizontal strut 10 ft. long, having pin-joints at its ends, is of rectangular cross-section, $1\frac{1}{2}$ in. wide by 4 in. deep, and carries an axial thrust of 20,000 lb., together with a total vertical load of 2000 lb.

uniformly distributed. Assuming that there is no friction at the pins, estimate the maximum stress induced. $E = 30 \times 10^6$ lb. per sq. in.

Apply the formula

$$f = \frac{w}{n^2}(\sec \tfrac{1}{2}nL - 1)\frac{m_c}{I} + \frac{P}{A},$$

where $n^2 = \dfrac{P}{EI}$,

$w = \frac{2000}{120}$ lb. per in.,

$P = 20{,}000$ lb.,

$E = 30 \times 10^6$ lb. per sq. in.,

$I = \frac{1}{12} \times 1\frac{1}{2} \times 4^3 = 8$ in.^{4},

$L = 120$ in.,

$m_c = 2$ in.,

$A = 6$ sq. in.,

$$n^2 = \frac{P}{EI} = \frac{20{,}000}{30 \times 10^6 \times 8} = \frac{1}{120 \times 10^2},$$

$$n = \frac{1}{109 \cdot 5}.$$

$$\therefore f = \frac{2000 \times 120 \times 10^2}{120}\left[\sec \frac{60}{109 \cdot 5} - 1\right]\frac{2}{8} + \frac{20{,}000}{6}$$

$$= 200{,}000(1 \cdot 17 - 1)\frac{1}{4} + \frac{20{,}000}{6}$$

$$= 8500 + 3333$$

$$= 11{,}830 \text{ lb. per sq. in.}$$

5. Tie-bar with Lateral Load.

It is convenient at this stage to treat this problem, although it is not a strut example.

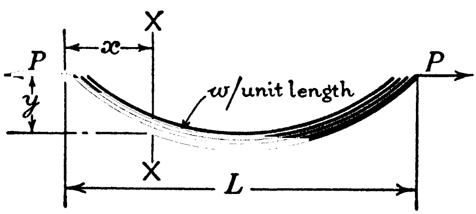


Fig. 10.11

Using the same notation as for the strut with lateral load, the bending moment at XX

$$= -Py + \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2.$$

$$\therefore\ EI\frac{d^2y}{dx^2} = -\{-Py + \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2\},$$

and $$\frac{d^2y}{dx^2} - n^2y + \frac{n^2wLx}{2P} - \frac{n^2wx^2}{2P} = 0, \text{ where } n^2 = \frac{P}{EI}.$$

The solution of this equation is

$$y = A\cosh nx + B\sinh nx + \frac{wLx}{2P} - \frac{wx^2}{2P} - \frac{w}{n^2P}.$$

The known conditions are $y = 0$ when $x = 0$,

and $$\frac{dy}{dx} = 0 \text{ when } x = \tfrac{1}{2}L.$$

Substituting for y and x,

$$y = 0 = A - \frac{w}{n^2P},$$

$$\therefore\ A = \frac{w}{n^2P}.$$

$$\frac{dy}{dx} = \frac{w}{n^2P}\,.\,n\sinh nx + nB\cosh nx + \frac{wL}{2P} - \frac{wx}{P}.$$

Substituting for dy/dx and x,

$$\frac{dy}{dx} = 0 = \frac{w}{nP}\sinh\tfrac{1}{2}nL + nB\cosh\tfrac{1}{2}nL + \frac{wL}{2P} - \frac{wL}{2P}.$$

$$\therefore\ B = -\frac{w}{n^2P}\tanh\tfrac{1}{2}nL.$$

$$y = \frac{w}{n^2P}\cosh nx - \frac{w}{n^2p}\tanh\tfrac{1}{2}nL\sinh nx + \frac{wLx}{2P} - \frac{wx^2}{2P} - \frac{w}{n^2P}.$$

The maximum deflection will occur when $x = \tfrac{1}{2}L$.

$$y_{\max} = \frac{w}{n^2P}\{\cosh\tfrac{1}{2}nL - \tanh\tfrac{1}{2}nL\sinh\tfrac{1}{2}nL\} + \frac{wL^2}{4P} - \frac{wL^2}{8P} - \frac{w}{n^2P}$$

$$= \frac{w}{n^2P}\{\operatorname{sech}\tfrac{1}{2}nL - 1\} + \frac{wL^2}{8P}.$$

The maximum bending moment will occur at the centre, and equals

$$-Py_{\max} + \tfrac{1}{2}wL \cdot \tfrac{1}{2}L - \tfrac{1}{2}w(\tfrac{1}{2}L)^2$$

$$= -P\left[\frac{w}{n^2P}(\operatorname{sech}\tfrac{1}{2}nL - 1) + \frac{wL^2}{8P}\right] + \tfrac{1}{4}wL^2 - \tfrac{1}{8}wL^2$$

$$= -\frac{w}{n^2}(\operatorname{sech}\tfrac{1}{2}nL - 1) - \tfrac{1}{8}wL^2 + \tfrac{1}{8}wL^2$$

$$= -\frac{wEI}{P}\left(\operatorname{sech}\tfrac{1}{2}L\sqrt{\frac{P}{EI}} - 1\right)$$

$$= \frac{wEI}{P}\left(1 - \operatorname{sech}\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right).$$

The maximum tensile stress will occur at the centre of span on the convex side.

Maximum tensile stress = direct stress + bending stress

$$= \frac{P}{A} + \frac{M_{\max}}{I} \times m_t$$

$$= \frac{P}{A} + \frac{wEm_t}{P}\left[1 - \operatorname{sech}\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right],$$

where A = area, and m_t = distance of farthest material from neutral surface on tensile side.

Maximum stress on concave side = direct stress − bending stress

$$= \frac{P}{A} - \frac{M_{\max}}{I} \times m_c$$

$$= \frac{P}{A} - \frac{wEm_c}{P}\left(1 - \operatorname{sech}\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right),$$

where m_c = distance of farthest material from neutral surface on compressive side. If this result is positive, the stress is tensile, but if negative, it is a compressive stress.

6. Struts of Variable Cross-section.

A strut of uniform cross-section is not the most economical form to use. In the case of a strut with free ends it is obvious that some of the material could be removed from the two ends without impairing its stability, whilst a strut fixed at one end and free at the other does not require so much material at the free end. This reduction in area is very important where it is essential to keep the weight of the member as low as possible, e.g. in the case of aeroplane structures.

An abrupt change in section, as shown in fig. 10.12, is a simple case.

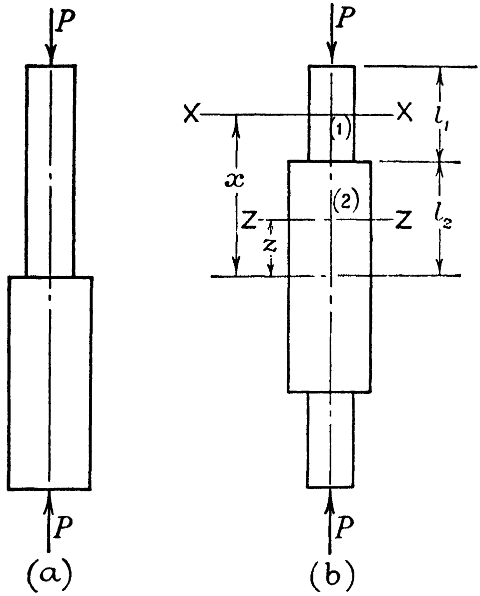


Fig. 10.12

Using a free-ended strut with change in section, as shown in fig. 10.12(*b*), we can effect a solution if we make the same assumption as we made in the derivation of Euler's formula.

Second moment for section (1) $= I_1$.

Second moment for section (2) $= I_2$.

For section XX, the bending moment is $P(\delta - y)$, where δ is the central deflection (see fig. 10.13).

$$\therefore\ EI\frac{d^2y}{dx^2} = +P(\delta - y).$$

$$\therefore\ \frac{d^2y}{dx^2} + n_1^2y - n_1^2\delta = 0, \text{ where } n_1^2 = \frac{P}{EI_1}.$$

The solution of this is

$$y_1 = A\cos n_1x + B\sin n_1x + \delta.$$

Known conditions: $y_1 = \delta$ when $x = L$.

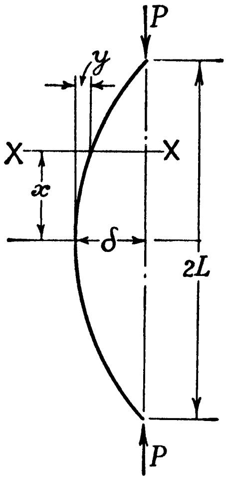


Fig. 10.13

$$\therefore\ y_1 = \delta = A\cos n_1L + B\sin n_1L + \delta,$$
$$\therefore\ 0 = A\cos n_1L + B\sin n_1L.$$
$$\therefore\ A = -B\tan n_1L.$$
$$\therefore\ y_1 = -B\tan n_1L\cos n_1x + B\sin n_1x + \delta, \quad . \ . \quad (1)$$

and $$\frac{dy_1}{dx} = +\,n_1B\tan n_1L\sin n_1x + n_1B\cos n_1x. \quad . \ . \quad (2)$$

For section ZZ, the bending moment $= P(\delta - y)$ (see figs. 10.12*b* and 10.13).

$$\therefore\ EI\frac{d^2y_2}{dz^2} = P(\delta - y).$$

The solution of this is

$$y_2 = C\cos n_2z + D\sin n_2z + \delta,\ \text{where } n_2{}^2 = \frac{P}{EI_2}.$$

Known conditions:

$$y_2 = 0 \text{ when } z = 0,\ \therefore\ C = -\delta,$$

and $$\frac{dy_2}{dz} = 0 \text{ when } z = 0.$$

$$\therefore\ \frac{dy_2}{dz} = 0 = \delta n_2\sin(n_2 \,.\, 0) + n_2D\cos(n_2 \,.\, 0)$$
$$= 0 + D,\ \therefore\ D = 0.$$
$$\therefore\ y_2 = -\delta\cos n_2z + \delta, \quad . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

and $$\frac{dy_2}{dz} = n_2\delta\sin n_2z. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4)$$

The deflection when $x = z = l_2$ is given by equations (1) and (3), and is

$$y_1 = -B \tan n_1 L \cos n_1 l_2 + B \sin n_1 l_2 + \delta$$

$$= y_2 = -\delta \cos n_2 l_2 + \delta.$$

$$\therefore\ B[\tan n_1 L \cos n_1 l_2 - \sin n_1 l_2] = \delta \cos n_2 l_2,$$

$$B[\sin n_1 L \cos n_1 l_2 - \sin n_1 l_2 \cos n_1 L] = \delta \cos n_2 l_2 \cos n_1 L,$$

i.e.

$$B[\sin (n_1 L - n_1 l_2)] = \delta \cos n_2 l_2 \cos n_1 L,$$

or

$$B = \frac{\delta \cos n_2 l_2 \cos n_1 L}{\sin n_1 l_1}.$$

Equating the slope when $x = z = l_2$, equations (2) and (4) give

$$\frac{n_1 \delta \cos n_2 l_2 \cos n_1 L}{\sin n_1 l_1} \tan n_1 L \sin n_1 l_2 + \frac{n_1 \delta \cos n_2 l_2 \cos n_1 L \cos n_1 l_2}{\sin n_1 l_1} = n_2 \delta \sin n_2 l_2.$$

$$\therefore\ n_1 \delta \frac{\cos n_2 l_2}{\sin n_1 l_1} \left[\cos n_1 L \frac{\sin n_1 L}{\cos n_1 L} \sin n_1 l_2 + \cos n_1 l_2 \cos n_1 L\right] = n_2 \delta \sin n_2 l_2.$$

$$\therefore\ n_1 \delta \frac{\cos n_2 l_2}{\sin n_1 l_1} [\cos (n_1 L - n_1 l_2)] = n_2 \delta \sin n_2 l_2.$$

$$\therefore\ n_1 \delta \frac{\cos n_2 l_2}{\sin n_1 l_1} \cos n_1 l_1 = n_2 \delta \sin n_2 l_2.$$

$$\therefore\ n_1 \delta \frac{\cos n_2 l_2}{\tan n_1 l_1} = n_2 \delta \sin n_2 l_2.$$

$$\therefore\ \frac{n_1}{n_2} = \frac{\sin n_2 l_2}{\cos n_2 l_2} \tan n_1 l_1,$$

i.e.

$$\frac{\sqrt{\frac{P}{EI_1}}}{\sqrt{\frac{P}{EI_2}}} = \tan l_2 \sqrt{\frac{P}{EI_2}} \tan l_1 \sqrt{\frac{P}{EI_1}}.$$

$$\therefore\ \sqrt{\frac{I_2}{I_1}} = \tan l_2 \sqrt{\frac{P}{EI_2}} \tan l_1 \sqrt{\frac{P}{EI_1}}.$$

When the ratio I_2 to I_1 is known, and the values of l_1 and l_2, a solution may be obtained by trial and error.

Example 6.—Find the buckling load for a strut 10 ft. in length, fixed at one end and free at the other, which is made of two tubes. The lower tube is 8 ft. long and 6 in. external diameter and the upper tube is

2 ft. long and 4 in. external diameter, whilst the thickness of metal in each tube is $\frac{1}{4}$ in. $E = 13{,}000$ tons per sq. in.

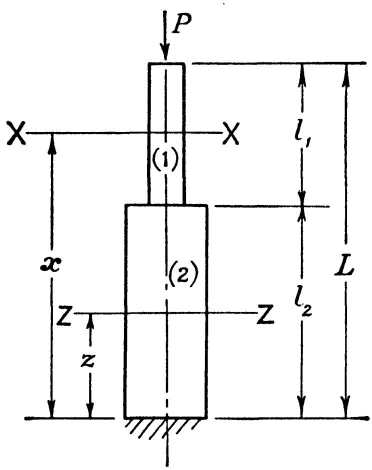


Fig. 10.14

For the lower tube, the second moment

$$I_2 = \tfrac{1}{64}\pi[6^4 - (5\tfrac{1}{2})^4] = 18{\cdot}69 \text{ in.}^4 \text{ units.}$$

For the upper tube, the second moment

$$I_1 = \tfrac{1}{64}\pi[4^4 - (3\tfrac{1}{2})^4] = 5{\cdot}20 \text{ in.}^4 \text{ units.}$$

Now $$\sqrt{\frac{I_2}{I_1}} = \tan l_2 \sqrt{\frac{P}{EI_2}} \tan l_1 \sqrt{\frac{P}{EI_1}},$$

$$\therefore \sqrt{\frac{18{\cdot}69}{5{\cdot}20}} = \tan 96 \times \sqrt{\frac{P}{13{,}000 \times 18{\cdot}69}} \tan 24 \times \sqrt{\frac{P}{13{,}000 \times 5{\cdot}20}}.$$

$$\therefore 1{\cdot}895 = \tan(0{\cdot}195\sqrt{P}) \tan(0{\cdot}092\sqrt{P}).$$

By trial and error, $P = 40$.

For a similar strut, but one having the diameter uniform throughout its length and equal to the larger diameter, the buckling load would be

$$P = \frac{\pi^2 EI}{4L^2}$$

$$= \frac{\pi^2 \times 13{,}000 \times 18{\cdot}69}{4 \times 120^2}$$

$$= 41{\cdot}69 \text{ tons.}$$

The ratio of the weights of the uniform and non-uniform struts

$$= \frac{96 \times \frac{1}{4}\pi[6^2 - (5\frac{1}{2})^2] + 24 \times \frac{1}{4}\pi[4^2 - (3\frac{1}{2})^2]}{120 \times \frac{1}{4}\pi[6^2 - (5\frac{1}{2})^2]}$$

$$= \frac{8 \times 5{\cdot}75 + 2 \times 3{\cdot}75}{10 \times 5{\cdot}75}$$

$$= 0{\cdot}93.$$

Whilst the ratio of the buckling loads

$$= \frac{40}{41{\cdot}69} = 0{\cdot}96.$$

EXAMPLES

1. A strut of length l is encastered at one end, its other end being supported in such a way that, if any transverse displacement δ occurs there, a transverse restraining force $\mu\delta$ acts on the strut at that end. Show that the buckling load P is given by the solution of the equation

$$\tan nl = nl - \frac{nP}{\mu},$$

where

$$n^2 = \frac{P}{EI}.$$

2. A straight uniform strut of length l has hinged ends, and is further supported at its mid-point in such a way as to permit angular rotation but to prevent lateral displacement. It is subjected to an axial thrust P, applied at its ends, together with a couple M, applied at its mid-point, and in the weakest plane of bending. Obtain an expression for the centre line of the strut in terms of M and P, and show that, for small values of M, the critical load is given by

$$\frac{P}{EI} = \frac{4\pi^2}{l^2},$$

where E is Young's modulus, and I is the least moment of inertia of the cross-section.

3. Establish Euler's formula for the buckling load on a long slender strut when axially loaded. Hence show how Rankine's formula for columns of medium length may be obtained from a consideration of the crushing load for a short column and Euler's equation for a long one.

4. A rolled-steel section 9 in. × 7 in. has an area of 14·7 sq. in., and its least radius of gyration is 1·65 in. If used as a strut 12 ft. long, with ends supported but not fixed in direction, determine the buckling load upon it. [153 tons.]

5. A tubular steel strut is of $2\frac{1}{2}$ in. external and 2 in. internal diameter. It is 8 ft. long and has hinged ends. The load is parallel to the axis but

eccentric to it. Find the maximum allowable eccentricity for a crippling load 0·75 × (Euler load), the yield stress being 45,000 lb. per sq. in. Work from first principles. (London Univ.) [0·902 in.]

6. By means of the Rankine formula, the buckling load for a hollow, cylindrical pillar, 20 ft. long, 8 in. external diameter, 6 in. internal diameter, is found to be 324·6 tons. The direct crushing strength of the material is taken as 36 tons per sq. in., and the ends considered fixed. What would be the buckling load of the same pillar, shortened by 5 ft., under the same conditions? [436 tons.]

7. A compound pillar, 36 ft. long, with free ends, is built up from two 14 in. × $\frac{1}{2}$ in. steel plates and two 10 in. × 3 in. channels. Find the crippling load from the Rankine formula, taking the constants f_c and a as 21 tons per sq. in. and $\frac{1}{7500}$ respectively. For a 10 in. × 3 in. channel, $I_{XX} = 83$ in.^{4}, $I_{YY} = 4$ in.^{4}, $A = 5{\cdot}67$ sq. in., and the distance of the centre of area from the back of the channels is 0·74 in. [232 tons.]

8. (*a*) Calculate the crippling stress, using the Euler formula, for a pin-ended strut 6 ft. long consisting of a tube of 2·5 in. outside diameter and wall thickness 0·125 in. $E = 13{,}450$ lb. per sq. in.

In a compression test of a short length of this tube a load of 31 tons caused failure, and when tested as a strut with rounded ends, 6 ft. long, it failed at 15·6 tons. Find the value of the constant a in the Rankine formula. [18·2; $\frac{1}{7327}$.]

9. A straight strut of uniform section and length L is subjected to an axial thrust P and a uniformly distributed transverse load of intensity w per unit of its length. Show that the maximum bending moment at the central section is given by the formula

$$M = \frac{wEI}{P}\left\{\sec\left(\tfrac{1}{2}L\sqrt{\frac{P}{EI}}\right) - 1\right\}.$$

The strut is to be taken as pin-jointed at its ends.

10. A B.S.B., 12 in. × 5 in. and 20 ft. long, is used as a strut with hinged ends. It carries an axial load of 30 tons together with a lateral load of $\frac{1}{2}$ ton per ft. uniformly distributed along one flange over its entire length. Determine the maximum stress produced. $I = 220{\cdot}1$ in.^{4}; $A = 9{\cdot}41$ sq. in.; $E = 12{,}500$ tons per sq. in. (London Univ.)
[11·86 tons per sq. in.]

11. A locomotive coupling rod, 8 ft. 6 in. between centres, is of symmetrical I section. The outside dimensions are $1\frac{1}{2}$ in. wide by $4\frac{1}{2}$ in. deep; web, $\frac{1}{2}$ in. thick, and flanges each $\frac{3}{4}$ in. thick. Find the maximum compressive stress in the rod due to an end thrust of 13·5 tons and a uniform transverse load of 300 lb. per ft. $E = 13{,}000$ tons per sq. in. (London Univ.)
[7·68 tons per sq. in.]

CHAPTER XI

Stresses in Curved Bars

The expressions $$\frac{M}{I} = \frac{f}{y} = \frac{E}{\rho}$$

were deduced for beams in which the radius of curvature was large compared with the cross-sectional dimensions of the beam. In the case of chain hooks and links, the radius of curvature is comparable with the cross-sectional dimensions.

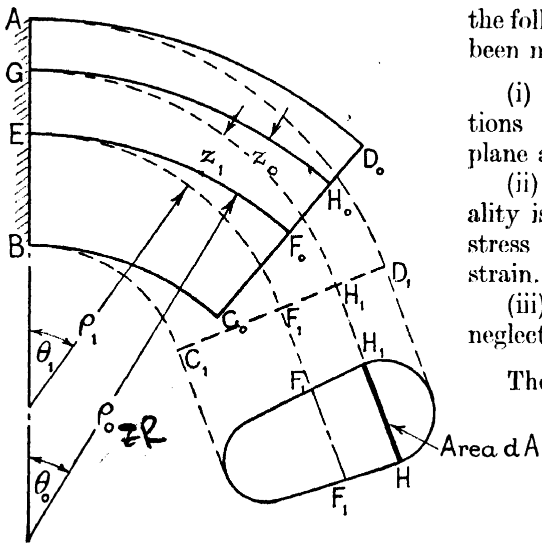


Fig. 11.1

In the following solution the following assumptions have been made:

(i) Plane transverse sections before bending remain plane after bending.

(ii) The limit of proportionality is not exceeded, i.e. the stress is proportional to the strain.

(iii) The radial stress is neglected.

The above assumptions are the ones used by Winkler in developing the theory of the strength of crane hooks. The problem of stress distribution for the case of plane stress can be solved by the theory of elasticity. A deep rectangular section falls approximately in the category of plane stress, where the stress distribution consists of radial stresses, circumferential stresses and shear stresses. It can be shown that the radial stress is small compared with the circumferential stress.

Consider a small portion of a curved beam. Before straining the portion is ABC_0D_0, whilst after straining it is ABC_1D_1. Let the central

axis be EF_0 before straining and EF_1 after straining, and ρ_0 and ρ_1 the radii of curvature respectively. Let z_0 and z_1 be the distance of any strip GH_0 and GH_1 from the central axes, before and after straining respectively.

For the strip EF_0, we have

$$\theta_0\rho_0 = EF_0 = s_0 \quad \text{(say)}.$$

$$\therefore\ \theta_0 = \frac{s_0}{\rho_0}, \qquad \ldots \qquad (a)$$

and

$$\theta_1\rho_1 = EF_1 = s_1 \quad \text{(say)}.$$

If e_0 is the strain on EF_0 when it is increased to EF_1, then

$$s_1 = s_0 + s_0e_0 = s_0(1 + e_0),$$

and

$$\theta_1\rho_1 = s_0(1 + e_0),$$

$$\therefore\ \theta_1 = \frac{s_0(1 + e_0)}{\rho_1}. \qquad \ldots \qquad (b)$$

For the strip GH_0, $\qquad \theta_0(\rho_0 + z_0) = GH_0,$

and $\qquad \theta_1(\rho_1 + z_1) = GH_1.$

$$\text{The strain on } GH_0 = e = \frac{GH_1 - GH_0}{GH_0}$$

$$= \frac{\theta_1(\rho_1 + z_1) - \theta_0(\rho_0 + z_0)}{\theta_0(\rho_0 + z_0)}$$

$$= \frac{\theta_1(\rho_1 + z_1)}{\theta_0(\rho_0 + z_0)} - 1.$$

Substituting for θ_0 and θ_1 from (a) and (b),

$$e = \frac{\dfrac{s_0}{\rho_1}(1 + e_0)(\rho_1 + z_1)}{\dfrac{s_0}{\rho_0}(\rho_0 + z_0)} - 1$$

$$= \frac{(1 + e_0)\left(1 + \dfrac{z_1}{\rho_1}\right)}{\left(1 + \dfrac{z_0}{\rho_0}\right)} - 1.$$

Assuming there is no radial stress, there will be no difference between z_0 and z_1. Let $z = z_0 = z_1$, then

$$e = \frac{(1 + e_0)\left(1 + \frac{z}{\rho_1}\right)}{\left(1 + \frac{z}{\rho_0}\right)} - 1.$$

The stress on $GH_0 = f = eE$, where E = modulus of elasticity,

$$\therefore f = E\left\{\frac{(1 + e_0)\left(1 + \frac{z}{\rho_1}\right)}{1 + \frac{z}{\rho_0}} - 1\right\}$$

$$= E\left\{\frac{(1 + e_0)\left(\frac{z}{\rho_1} - \frac{z}{\rho_0}\right)}{1 + \frac{z}{\rho_0}} + e_0\right\}. \quad . \quad . \quad (1)$$

The total stress on the section $= F = \int Ee\,dA$, where dA is an element of the area of cross-section H_1H_1.

$$\therefore F = \int E\left[\frac{(1 + e_0)\left(\frac{z}{\rho_1} - \frac{z}{\rho_0}\right)}{\left(1 + \frac{z}{\rho_0}\right)} + e_0\right] dA$$

$$= \int \frac{E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)}{\left(1 + \frac{z}{\rho_0}\right)} z\,dA + Ee_0A,$$

where A = cross-sectional area of the beam,

$$= E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\int \frac{z\,dA}{1 + \frac{z}{\rho_0}} + Ee_0A. \quad . \quad (2)$$

The moment of stress on the elemental area at $H_1H_1 = Eez\,dA$.

$$\therefore \text{ total moment } M = \int E\left[\frac{(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)z}{\left(1 + \frac{z}{\rho_0}\right)} + e_0\right] z\,dA.$$

Now z is the distance of the strip from the central axis, therefore $\int z\,dA$ is the sum of the moments of the areas about the central axis, which is zero. Therefore $\int Ee_0 z\,dA = 0$.

$$\therefore\ M = \int \frac{E(1+e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)}{\left(1+\frac{z}{\rho_0}\right)} z^2\,dA$$

$$= E(1+e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\int \frac{z^2\,dA}{1+\frac{z}{\rho_0}}. \quad . \quad . \quad (3)$$

The solutions of equations (2) and (3) will necessitate the solution of

$$\int \frac{z\,dA}{1+z/\rho_0} \text{ and } \int \frac{z^2\,dA}{1+z/\rho_0},$$

and their result will depend on the form of the cross-section of the beam. It is generally simpler to assume that

$$\int \frac{z^2\,dA}{1+z/\rho_0} = Ah^2,$$

where A is the area of cross-section, and h^2 a constant of the section.

Now
$$\int \frac{z\,dA}{1+\frac{z}{\rho_0}} = \int \frac{z\rho_0}{\rho_0+z}\,dA$$

$$= \int \left(z - \frac{z^2}{\rho_0+z}\right) dA$$

$$= \int z\,dA - \int \frac{z^2}{\rho_0+z}\,dA$$

$$= 0 - \frac{1}{\rho_0}\int \left(\frac{z^2}{1+\frac{z}{\rho_0}}\right) dA$$

$$= -\frac{Ah^2}{\rho_0}.$$

Substituting in equations (2) and (3),

$$F = E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\left(-\frac{Ah^2}{\rho_0}\right) + Ee_0A$$

$$= EA\left[e_0 - (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}\right]. \quad . \quad . \quad . \quad . \quad (4)$$

$$M = E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As it is assumed that plane sections before bending remain plane after bending, there cannot be any direct stress across the section. Therefore $\int Ee\,dA = 0$.

$$0 = EA\left[e_0 - (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}\right],$$

$$\therefore\ e_0 = (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}$$

$$= \frac{M}{EA\rho_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (1) gives the stress on an element at GH_0 and is

$$E\left\{\frac{(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)z}{\left(1 + \frac{z}{\rho_0}\right)} + e_0\right\}.$$

Substituting from (5), the stress is

$$f = \frac{M}{Ah^2} \cdot \frac{z}{1 + \frac{z}{\rho_0}} + Ee_0.$$

Substituting for e_0 from (6),

$$f = \frac{M}{Ah^2} \cdot \frac{z\rho_0}{\rho_0 + z} + \frac{M}{A\rho_0}$$

$$= \frac{M}{A\rho_0}\left(\frac{z}{\rho_0 + z} \cdot \frac{\rho_0^2}{h^2} + 1\right). \quad . \quad . \quad . \quad . \quad (7)$$

In the case considered the section is further from the centre of curvature than the neutral axis, hence the stress f will be tensile. If the section had been on the other side of the neutral surface, the value

of z would have been negative. A negative result for f would indicate a compression in the above form of bending.

It will be noticed from equation (7) that the distribution of stress is not linear but hyperbolic. The neutral axis does not pass through the centre of area of the section, but is displaced towards the intrados through a radial distance of $\rho_0 h^2/(\rho_0^2 + h^2)$.

Values of h^2.

Before applying the above equations to particular problems it will be beneficial to ascertain the values of h^2 for a number of standard shapes.

(i) *Rectangular section* (breadth B and depth D).

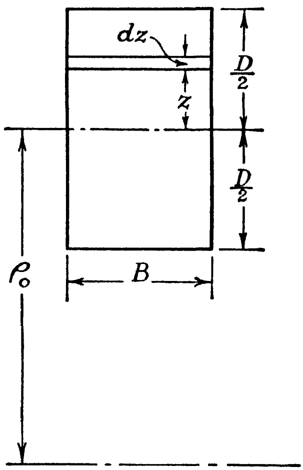


Fig. 11.2

$$\frac{Ah^2}{\rho_0} = \int \frac{z\,dA}{1 + z/\rho_0}$$

$$\frac{Ah^2}{\rho_0} = \int_{-\frac{1}{2}D}^{+\frac{1}{2}D} \frac{\rho_0 z B\,dz}{\rho_0 + z}$$

$$= B\rho_0 \int_{-\frac{1}{2}D}^{+\frac{1}{2}D} \left(1 - \frac{\rho_0}{\rho_0 + z}\right) dz.$$

$$\therefore\ h^2 = \frac{\rho_0^3}{D} \log_e \frac{2\rho_0 + D}{2\rho_0 - D} - \rho_0^2, \quad . \quad . \quad . \quad . \quad (8)$$

or by expansion,

$$h^2 = \rho_0^2 \left\{ \tfrac{1}{12}\left(\frac{D}{\rho_0}\right)^2 + \tfrac{1}{80}\left(\frac{D}{\rho_0}\right)^4 + \tfrac{1}{448}\left(\frac{D}{\rho_0}\right)^6 + \ldots \right\}. \quad . \quad (9)$$

(ii) *Circular section* (diameter D).

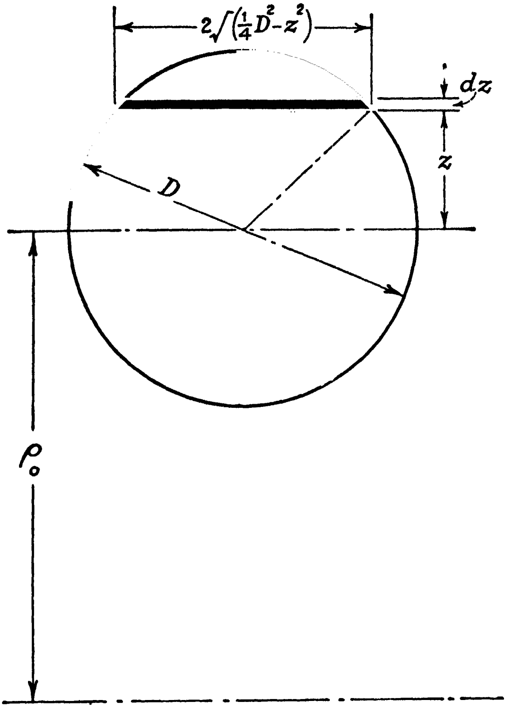


Fig. 11.3

$$-\frac{Ah^2}{\rho_0}=\int_{-\frac{1}{2}D}^{+\frac{1}{2}D}\rho_0\frac{2z\sqrt{(\frac{1}{4}D^2-z^2)}\,dz}{\rho_0+z}$$

$$=\rho_0\int_{-\frac{1}{2}D}^{+\frac{1}{2}D}\frac{zx\,dz}{\rho_0+z},\ \text{where } x=2\sqrt{(\tfrac{1}{4}D^2-z^2)}$$

$$=-\rho_0^2\int_{-\frac{1}{2}D}^{+\frac{1}{2}D}\frac{x}{\rho_0+z}\,dz+\int_{-\frac{1}{2}D}^{+\frac{1}{2}D}\rho_0 x\,dz$$

$$=-\tfrac{1}{2}\rho_0^2D^2\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\frac{\cos^2\theta}{\frac{1}{2}D\sin\theta+\rho_0}\,d\theta+\tfrac{1}{4}\pi\rho_0D^2$$

$$=-2\pi\rho_0^3+2\pi\rho_0^2\sqrt{[\rho_0^2-(\tfrac{1}{2}D)^2]}+\tfrac{1}{4}\pi\rho_0D^2.$$

$$\therefore\ h^2=\rho_0^2\left\{2\left(\frac{2\rho_0}{D}\right)^2-\frac{8\rho_0}{D^2}\sqrt{(\rho_0^2-\tfrac{1}{4}D^2)}-1\right\},$$

or by expansion,

$$h^2=\frac{D^2}{16}\left\{1+\tfrac{1}{2}\left(\frac{D}{2\rho_0}\right)^2+\tfrac{5}{16}\left(\frac{D}{2\rho_0}\right)^4+\ldots\right\}. \quad . \quad (10)$$

(iii) *Trapezium section.*

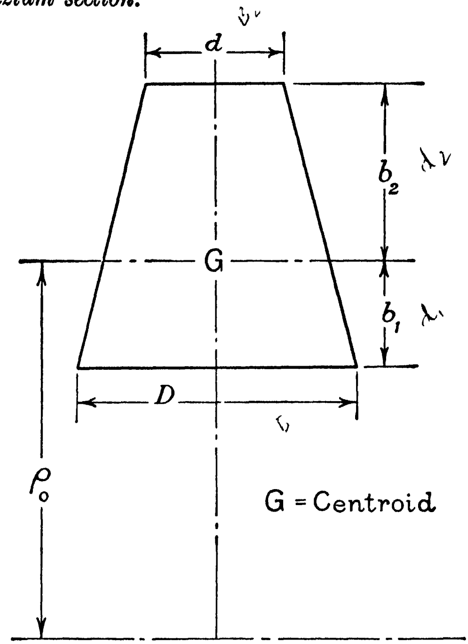


Fig. 11.4

$$h^2 = \rho_0^2\left[\frac{\rho_0}{A}\left\{\left[d + \frac{D-d}{b_1+b_2}(b_1+\rho_0)\right]\log_e\frac{\rho_0+b_1}{\rho_0-b_2} - (D-d)\right\} - 1\right], \quad (11)$$

where A is the area of section.

For other sections it will generally be necessary to integrate graphically.

Example 1.—The mean radius of curvature of a 2 in. square bar is 3 in. If the bar is subjected to a bending moment of 5000 lb.-in., which is tending to reduce the radius of curvature, find the maximum stresses in the material.

Apply formula $\quad f = \frac{M}{A\rho_0}\left(\frac{z}{\rho_0+z}\cdot\frac{\rho_0^2}{h^2}+1\right),$

where $\quad M = +5000$ lb.-in.,

$A = 4$ sq. in.,

$\rho_0 = 3$ in.,

$z = -1$ for inner edge and $+1$ for outer edge.

$h^2 = \frac{\rho_0^3}{D}\log_e\frac{2\rho_0+D}{2\rho_0-D} - \rho_0^2$, where $D = 2$ in.

$$\therefore\ h^2 = \frac{3^3}{2}\log_e\frac{2\times 3+2}{2\times 3-2} - 3^2$$

$$= \frac{27}{2}\log_e 2 - 9$$

$$= 13{\cdot}5 \times 0{\cdot}693 - 9$$

$$= 9{\cdot}36 - 9$$

$$= 0{\cdot}36.$$

For inner edge, $f = \frac{5000}{4\times 3}\left(\frac{-1}{3-1}\cdot\frac{3^2}{0{\cdot}36} + 1\right)$

$$= \tfrac{5000}{12}(-12{\cdot}5 + 1)$$

$$= -\ 4790 \text{ lb. per sq. in. compression.}$$

For outer edge, $f = \frac{5000}{4\times 3}\left(\frac{+1}{3+1}\cdot\frac{3^2}{0{\cdot}36} + 1\right)$

$$= \tfrac{5000}{12}(6{\cdot}25 + 1)$$

$$= 3020 \text{ lb. per sq. in. tension.}$$

1. Crane Hooks.

The formula deduced for curved beams may be applied to crane hooks. The most highly stressed section will be *ab*. In addition to the bending stress caused by the bending moment *Wl*, there will be a direct tensile stress due to the load *W*. The net stress will be the algebraic sum of the two stresses.

The bending stress is given by

$$f = \frac{M}{A\rho_0}\left(\frac{z}{\rho_0+z}\cdot\frac{\rho_0^2}{h^2}+1\right),$$

in which *M* is the bending moment and equals *Wl*. It is negative in this case, because it is tending to increase the radius of curvature, whilst the above formula was deduced for a case where the radius of curvature was reduced. The values of h^2 and *A* depend on the section of the material.

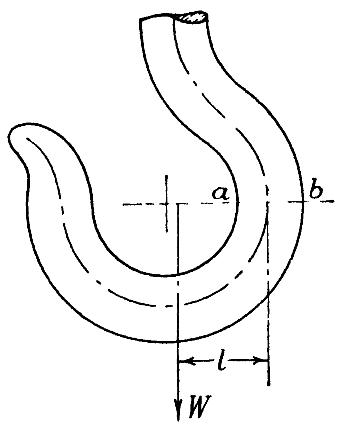


Fig. 11.6

The net stress at *a* and *b* is

$$\frac{M}{A\rho_0}\left(\frac{z}{\rho_0+z}\cdot\frac{\rho_0^2}{h^2}+1\right)+\frac{W}{A}. \quad . \quad . \quad . \quad . \quad (12)$$

Example 2.—A crane hook is trapezoidal in cross-section and carries a load of 2 tons which acts through the centre of curvature of the hook and at 3 in. from the centroid of the section on which the stress is a maximum. The inner width of the material is 2 in. and the outer width 1 in., whilst the depth is $2\frac{1}{4}$ in. The centroid of the section is therefore 1 in. from the inner edge. Find the maximum stresses in the material.

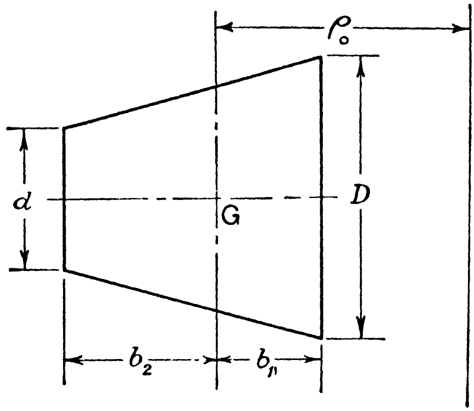


Fig. 116

Apply formula $f = \frac{M}{A\rho_0}\left(\frac{z}{\rho_0 + z}\cdot\frac{\rho_0^2}{h^2} + 1\right) + \frac{W}{A}$,

where $M = -Wl = -2 \times 3 = -6$ tons-in.,

$A = \frac{1}{2}(2 + 1) \times 2\frac{1}{4} = 3\frac{3}{8}$ sq. in.,

$\rho_0 = 3$ in.

$$h^2 = \rho_0^2\left[\frac{\rho_0}{A}\left\{\left[d + \frac{D - d}{b_1 + b_2}(b_1 + \rho_0)\right]\log_e\frac{\rho_0 + b_1}{\rho_0 - b_2} - (D - d)\right\} - 1\right],$$

where $d = 1$ in., $D = 2$ in., $b_1 = 1$ in., $b_2 = 1\frac{1}{4}$ in.

$$\therefore h^2 = 3^2\left[\frac{3}{3\frac{3}{8}}\left\{\left[1 + \frac{2 - 1}{1 + 1\frac{1}{4}}(1 + 3)\right]\log_e\frac{3 + 1}{3 - 1\frac{1}{4}} - (2 - 1)\right\} - 1\right]$$

$$= 9\left[\tfrac{24}{27}\left\{\left[1 + \tfrac{16}{9}\right]\log_e\tfrac{16}{7} - 1\right\} - 1\right]$$

$$= 9\left[\tfrac{24}{27}\left\{\left[1 + \tfrac{16}{9}\right] \times \cdot 827 - 1\right\} - 1\right]$$

$$= 1\cdot 395,$$

$z = -1$ for inner edge and $= +1\frac{1}{4}$ for outer edge. The positive or negative sign indicates whether the section is farther from or nearer to the centre of curvature than the neutral axis.

For inner edge,

$$f = \frac{-6}{3\frac{3}{8} \times 3}\left(\frac{-1}{3-1} \times \frac{3^2}{1{\cdot}395} + 1\right) + \frac{2}{3\frac{3}{8}}$$

$$= -\tfrac{16}{27}\left(-\tfrac{1}{2} \times \frac{9}{1{\cdot}395} + 1\right) + \tfrac{16}{27}$$

$$= 1{\cdot}318 + 0{\cdot}592$$

$$= 1{\cdot}91 \text{ tons per sq. in. tension.}$$

For outer edge,

$$f = \frac{-6}{3\frac{3}{8} \times 3}\left(\frac{1\frac{1}{4}}{3+1\frac{1}{4}} \times \frac{3^2}{1{\cdot}395} + 1\right) + \frac{2}{3\frac{3}{8}}$$

$$= -\tfrac{16}{27}(1{\cdot}90 + 1) + \tfrac{16}{27}$$

$$= -1{\cdot}71 + 0{\cdot}592$$

$$= -1{\cdot}12 \text{ tons per sq. in. compression.}$$

2. Rings.

A circular ring of mean radius ρ_0, subjected to a pull W at each end of the diameter, is shown in fig. 11.7. The centroids of the cross-section

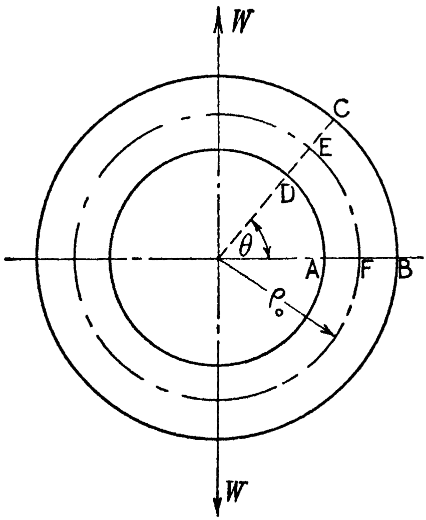


Fig. 11.7

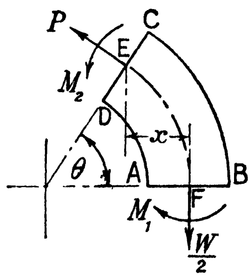


Fig. 11.8

lie on the dotted circle. If the ring is cut at AB and CD, then the forces $\frac{1}{2}W$ and P, and the couples M_1 and M_2, must act on AB and CD to keep the portion ABCD in the same state of equilibrium (fig. 11.8).

Taking moments about E,

$$M_2 = M_1 + \tfrac{1}{2}Wx$$

$$= M_1 + \tfrac{1}{2}W(\rho_0 - \rho_0 \cos\theta)$$

$$= M_1 + \tfrac{1}{2}W\rho_0(1 - \cos\theta), \quad . \quad . \quad . \quad . \quad (13)$$

where the moments are positive when they tend to reduce the radius of curvature, i.e. increase the curvature of the ring.

$$\text{The force on DC} = P = \tfrac{1}{2}W\cos\theta. \quad . \quad . \quad . \quad (14)$$

But the bending moment at any section of a curved beam is given by equation (5), i.e.

$$M = E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2,$$

and the force by equation (4), i.e.

$$F = EA\left\{e_0 - (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}\right\}.$$

Hence, substituting in (13),

$$E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2 = M_2$$
$$= M_1 + \tfrac{1}{2}W\rho_0(1 - \cos\theta), \quad . \quad (15)$$

and substituting in (14),

$$EA\left\{e_0 - (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}\right\} = P = \tfrac{1}{2}W\cos\theta. \quad . \quad (16)$$

Substituting in (16) for $E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2$ from (15),

$$EAe_0 - \frac{M_2}{\rho_0} = \tfrac{1}{2}W\cos\theta.$$

$$\therefore\ EAe_0 - \frac{M_1 + \tfrac{1}{2}W\rho_0(1 - \cos\theta)}{\rho_0} = \tfrac{1}{2}W\cos\theta.$$

$$\therefore\ EAe_0 - \frac{M_1}{\rho_0} - \tfrac{1}{2}W + \tfrac{1}{2}W\cos\theta = \tfrac{1}{2}W\cos\theta,$$

$$\therefore\ EAe_0 = \frac{M_1}{\rho_0} + \tfrac{1}{2}W. \quad . \quad (17)$$

Since M_1, W, ρ_0, E and A are constants, then e_0 is constant.

The change of slope of the curve of flexure $= \dfrac{M\,ds}{EI}$

$$= \frac{M}{EI}\rho_0\,d\theta,$$

where $d\theta$ is the angular change.

Hence from (15),

$$E(1+e_0)\left(\frac{1}{\rho_1}-\frac{1}{\rho_0}\right)Ah^2\,.\,\frac{\rho_0\,d\theta}{EI} = \left\{M_1 + \tfrac{1}{2}W\rho_0(1-\cos\theta)\right\}\frac{\rho_0\,d\theta}{EI}.$$

For the complete quadrant,

$$E(1+e_0)\left(\frac{1}{\rho_1}-\frac{1}{\rho_0}\right)Ah^2\rho_0\int_0^{\frac{1}{2}\pi}d\theta$$

$$= M_1\rho_0\int_0^{\frac{1}{2}\pi}d\theta + \tfrac{1}{2}W\rho_0^2\int_0^{\frac{1}{2}\pi}(1-\cos\theta)\,d\theta. \quad . \quad (18)$$

After straining, the length of the axis EF is

$$\rho_1\theta_1 = \int_0^{\theta_1}(1+e_0)\rho_0\,d\theta.$$

For the complete quadrant, $\theta_1 = \frac{1}{2}\pi$,

$$\therefore\ \rho_1\,.\,\tfrac{1}{2}\pi = \int_0^{\frac{1}{2}\pi}(1+e_0)\rho_0\,d\theta.$$

$$\therefore\ \int_0^{\frac{1}{2}\pi}(1+e_0)\frac{\rho_0}{\rho_1}\,d\theta = \tfrac{1}{2}\pi. \quad . \quad . \quad . \quad . \quad (19)$$

Now equation (18) can be expanded to

$$E(1+e_0)Ah^2\frac{\rho_0}{\rho_1}\int_0^{\frac{1}{2}\pi}d\theta - E(1+e_0)Ah^2\int_0^{\frac{1}{2}\pi}d\theta$$

$$= M_1\rho_0\int_0^{\frac{1}{2}\pi}d\theta + \tfrac{1}{2}W\rho_0^2\int_0^{\frac{1}{2}\pi}(1-\cos\theta)\,d\theta.$$

Substituting from (19),

$$EAh^2\,.\,\tfrac{1}{2}\pi - EAh^2\,.\,\tfrac{1}{2}\pi\frac{\rho_1}{\rho_0} = M_1\rho_0\,.\,\tfrac{1}{2}\pi + \tfrac{1}{2}W\rho_0^2\,.\,\tfrac{1}{2}\pi - \tfrac{1}{2}W\rho_0^2.$$

But $$\frac{\rho_1}{\rho_0} = (1+e_0),$$

$$\therefore\ EAh^2\,.\,\tfrac{1}{2}\pi - EAh^2\,.\,\tfrac{1}{2}\pi(1+e_0) = M_1\rho_0\,.\,\tfrac{1}{2}\pi + \tfrac{1}{2}W\rho_0^2(\tfrac{1}{2}\pi - 1). \quad (20)$$

But from (17), $$EAe_0 = \frac{M_1}{\rho_0} + \tfrac{1}{2}W.$$

$$\therefore\ -\tfrac{1}{2}\pi h^2\left(\frac{M_1}{\rho_0} + \tfrac{1}{2}W\right) = M_1\rho_0\tfrac{1}{2}\pi + \tfrac{1}{2}W\rho_0^2 \cdot \tfrac{1}{2}\pi - \tfrac{1}{2}W\rho_0^2.$$

$$\therefore\ \tfrac{1}{2}M_1\pi\left(\frac{h^2}{\rho_0} + \rho_0\right) = \tfrac{1}{2}W\rho_0^2 - \tfrac{1}{4}W\rho_0^2\pi - \tfrac{1}{4}Wh^2\pi$$

$$= \tfrac{1}{2}W\rho_0^2 - \tfrac{1}{4}W\pi\rho_0\left(\rho_0 + \frac{h^2}{\rho_0}\right).$$

$$\therefore\ M_1 = \frac{W\rho_0^2}{\pi\left(\frac{h^2}{\rho_0} + \rho_0\right)} - \tfrac{1}{2}W\rho_0$$

$$= \frac{W\rho_0^3}{\pi(h^2 + \rho_0^2)} - \tfrac{1}{2}W\rho_0$$

$$= \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)} - 1\right]. \quad . \quad (21)$$

Substituting in (13) for M_1,

$$M_2 = \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)} - 1\right] + \tfrac{1}{2}W\rho_0(1 - \cos\theta)$$

$$= \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)} - \cos\theta\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The only variable is θ, therefore the maximum bending moment will occur when $\cos\theta = 0$, i.e. when $\theta = \pm\tfrac{1}{2}\pi$.

Then $$M_{max} = \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)}\right]$$

$$= \frac{W\rho_0^3}{\pi(h^2 + \rho_0^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From (17), $$e_0 = \frac{1}{EA}\left(\frac{M_1}{\rho_0} + \tfrac{1}{2}W\right).$$

Substituting for M_1 from (21),

$$e_0 = \frac{1}{EA}\left[\tfrac{1}{2}W\rho_0\left(\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)} - 1\right) \div \rho_0 + \tfrac{1}{2}W\right]$$

$$= \frac{1}{EA}\left[\tfrac{1}{2}W\left(\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)}\right)\right]$$

$$= \frac{W}{\pi EA} \cdot \frac{\rho_0^2}{h^2 + \rho_0^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For a curved beam the stress at any section is given by equation (1),

i.e. $$\text{stress} = eE = E\left[\frac{(1+e_0)\left(\frac{z}{\rho_1}-\frac{z}{\rho_0}\right)}{1+\frac{z}{\rho_0}}+e_0\right].$$

But from (15), $$E(1+e_0)\left(\frac{1}{\rho_1}-\frac{1}{\rho_0}\right)Ah^2 = M_2,$$

$$\therefore \text{ stress} = f_1 = \frac{\frac{M_2 z}{Ah^2}}{1+\frac{z}{\rho_0}} + Ee_0$$

$$= \frac{\frac{M_2 z}{Ah^2}}{1+\frac{z}{\rho_0}} + \frac{W}{\pi A}\cdot\frac{\rho_0^2}{h^2+\rho_0^2}.$$

Substituting for M_2 from (22),

$$f_1 = \frac{\frac{W\rho_0 z}{2Ah^2}\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}-\cos\theta\right]}{\left(1+\frac{z}{\rho_0}\right)} + \frac{W}{\pi A}\cdot\frac{\rho_0^2}{h^2+\rho_0^2}$$

$$= \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left(\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}-\cos\theta\right)\left(\frac{z}{\rho_0+z}\right)\right]. \quad (25)$$

The net stress at CD will be equal to the algebraic sum of the bending and direct stress, i.e.

$$f = f_1 + \frac{W\cos\theta}{2A}$$

$$= \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left(\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}-\cos\theta\right)\left(\frac{z}{\rho_0+z}\right)\right] + \frac{W\cos\theta}{2A}. \quad (26)$$

When $\theta = \frac{1}{2}\pi$,

$$f = \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left\{\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}\right\}\frac{z}{\rho_0+z}\right].$$

For the outside of the ring, $z = y_1$,

$$\therefore f = \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left\{\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}\right\}\frac{y_1}{\rho_0+y_1}\right], \quad (27)$$

and will be tensile in nature.

For the inside of the ring, $z = -y_2$,

$$\therefore f = \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left\{\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}\right\}\frac{(-y_2)}{\rho_0-y_2}\right]. \quad (28)$$

If the result is negative, then the stress will be compressive.

When $\theta = 0$, from equation (26), stress at outside of ring is

$$f = \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left\{\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - 1\right\}\frac{y_1}{\rho_0+y_1}\right] + \frac{W}{2A}. \quad (29)$$

If the result is negative, then the stress is compressive.

Stress at inside of ring is

$$f = \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left\{\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - 1\right\}\frac{(-y_2)}{\rho_0-y_2}\right] + \frac{W}{2A}. \quad (30)$$

A positive result will indicate a tensile stress.

Example 3. A ring of 6 in. mean diameter is made of steel of 1 in. diameter. If it is subjected to a diametrical pull of 2000 lb., find the maximum tensile and compressive stresses in the steel.

Apply formula (26), namely,

$$f = \frac{W}{A}\left[\frac{\rho_0^2}{\pi(h^2+\rho_0^2)} + \frac{\rho_0^2}{2h^2}\left\{\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \cos\theta\right\}\frac{z}{\rho_0+z}\right] + \frac{W\cos\theta}{2A},$$

where $W = 2000$ lb., $A = \frac{1}{4}\pi$ sq. in., $\rho_0 = 3$ in.

$$h^2 = \frac{D^2}{16}\left\{1 + \frac{1}{2}\left(\frac{D}{2\rho_0}\right)^2 + \frac{5}{16}\left(\frac{D}{2\rho_0}\right)^4 + \ldots\right\},$$

where $D = 1$ in.

$$\therefore h^2 = \frac{1}{16}\left\{1 + \frac{1}{2}\left(\frac{1}{6}\right)^2 + \frac{5}{16}\left(\frac{1}{6}\right)^4\right\} \text{ approx.}$$

$$= \frac{1}{16}(1 + 0{\cdot}01389 + 0{\cdot}00024)$$

$$= 0{\cdot}06338.$$

When $\theta = \frac{1}{2}\pi$,

$$f = \frac{2000 \times 4}{\pi}\left[\frac{9}{\pi(\cdot 06338 + 9)} + \frac{9}{2 \times \cdot 06338}\left\{\frac{18}{\pi(\cdot 06338 + 9)}\right\}\frac{z}{3 + z}\right]$$

$$= 2545\left[\cdot 316 + \{71 \times \cdot 632\}\frac{z}{3 + z}\right]$$

$$= 804 + 114{,}100\frac{z}{3 + z}.$$

For outside of ring, $z = +\frac{1}{2}$,

$$\therefore f = 804 + 114{,}100 \times \frac{\cdot 5}{3\cdot 5} = 17{,}100 \text{ lb. per sq. in. tension.}$$

For inside of ring, $z = -\frac{1}{2}$.

$$\therefore f = 804 + 114{,}100 \times \frac{(-\cdot 5)}{2\cdot 5} = -22{,}000 \text{ lb. per sq. in.}$$

(The negative result indicates compressive stress.)

When $\theta = 0$,

$$f = \frac{2000 \times 4}{\pi}\left[\frac{9}{\pi(\cdot 06338 + 9)} + \frac{9}{2 \times \cdot 06338}\left\{\frac{18}{\pi(\cdot 06338 + 9)} - 1\right\}\frac{z}{3 + z}\right] + \frac{2000}{2(\frac{1}{4}\pi)}$$

$$= 2545\left[\cdot 316 + 71(\cdot 632 - 1)\frac{z}{3 + z}\right] + 1273$$

$$= 804 - 66{,}500\frac{z}{3 + z} + 1273$$

$$= 2077 - 66{,}500\frac{z}{3 + z}.$$

For outside of ring, $z = +\frac{1}{2}$,

$$\therefore f = 2077 - 66{,}500 \times \frac{\cdot 5}{3\cdot 5}$$

$$= -7420 \text{ lb. per sq. in.}$$

(The negative sign indicates compression.)

For inside of ring, $z = -\frac{1}{2}$,

$$\therefore f = 2077 - 66{,}500 \times \frac{(-\cdot 5)}{2\cdot 5}$$

$$= +15{,}380 \text{ lb. per sq. in. tension.}$$

3. Deformation of Axis of Ring.

The change of slope of the curve of flexure of a curved beam when a bending moment M is applied is

$$d\phi = \frac{M\,ds}{EI},$$

whilst the horizontal or vertical movement of the section is $\frac{My\,ds}{EI}$, where $y =$ horizontal or vertical distance from the datum line. Compare the deflection for a straight beam, i.e. $\frac{Mx\,dx}{EI}$.

In the case of the ring shown in fig. 11.9, the distance $y = \rho_0 \cos\theta$ from the vertical datum line YY, and $x = \rho_0 \sin\theta$ from the horizontal datum line XX.

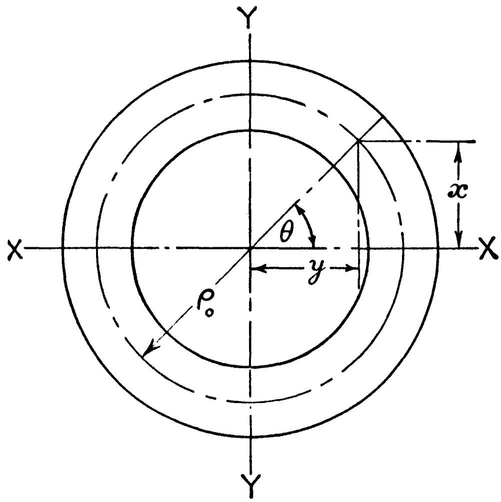


Fig. 11.9

If a pull acts along the axis YY, the total decrease in the horizontal radius

$$= \int_0^{\frac{1}{2}\pi} \frac{M\rho_0 \sin\theta\,ds}{EI}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{M\rho_0 \sin\theta\,\rho_0\,d\theta}{EI}.$$

Substituting the value for M in a curved beam from equation (22), i.e.

$$M_2 = \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \cos\theta\right],$$

total decrease in horizontal radius

$$= \int_0^{\frac{1}{2}\pi} \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \cos\theta\right]\frac{\rho_0^2}{EI}\sin\theta\,d\theta$$

$$= \frac{W\rho_0^3}{2EI}\left[\frac{2\rho_0^2(-\cos\theta)}{\pi(h^2+\rho_0^2)} + \tfrac{1}{4}\cos 2\theta\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{W\rho_0^3}{2EI}\left[0 - \tfrac{1}{4} + \frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \tfrac{1}{4}\right]$$

$$= \frac{W\rho_0^3}{2EI}\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \tfrac{1}{2}\right]. \qquad (31)$$

Total decrease in vertical radius $\int_0^{\frac{1}{2}\pi} M\rho_0\frac{\cos\theta\,ds}{EI}$

$$= \int_0^{\frac{1}{2}\pi} \tfrac{1}{2}W\rho_0\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \cos\theta\right]\frac{\rho_0^2}{EI}\cos\theta\,d\theta$$

$$= \frac{W\rho_0^3}{2EI}\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)}\sin\theta - \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{W\rho_0^3}{2EI}\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \tfrac{1}{4}\pi\right]. \qquad (32)$$

It will be noticed that equation (32) will be negative since $\frac{2\rho_0^2}{[\pi(h^2+\rho_0^2)]}$ has a maximum value of $2/\pi$ when h^2 is zero, and thus indicates an increase in the radius.

Example 4.—The mean diameter of a ring is 6 in., whilst the material of which it is constructed is of $1\frac{1}{2}$ in. diameter. If the ring is subjected to a compressive load of 2 tons along a diameter, find the decrease in diameter in the direction of the load, and the increase in diameter in the direction perpendicular to it. $E = 30 \times 10^6$ lb. per sq. in.

Applying equations (31) and (32), it must be remembered that these equations were deduced for a tensile load. The load will therefore be taken as -2 tons.

Assume the load is vertical, then, from (31), the horizontal change in radius

$$= \frac{W\rho_0^3}{2EI}\left[\frac{2\rho_0^2}{\pi(h^2+\rho_0^2)} - \tfrac{1}{2}\right],$$

where $W = -2 \text{ tons} = -2 \times 2240 \text{ lb.},$

$\rho_0 = 3 \text{ in.},$

$I = \frac{1}{64}\pi(\frac{3}{2})^4 = 0{\cdot}2485 \text{ in.}^4 \text{ units.}$

$$h^2 = \frac{D^2}{16}\left\{1 + \tfrac{1}{2}\left(\frac{D}{2\rho_0}\right)^2 + \tfrac{5}{16}\left(\frac{D}{2\rho_0}\right)^4 + \ldots\right\}$$

$$= \frac{(\frac{3}{2})^2}{16}\left\{1 + \tfrac{1}{2}\left(\frac{\frac{3}{2}}{2 \times 3}\right)^2 + \tfrac{5}{16}\left(\frac{\frac{3}{2}}{2 \times 3}\right)^4\right\} \text{ approx.}$$

$$= \tfrac{9}{64}\left\{1 + \tfrac{1}{32} + \tfrac{5}{4096}\right\}$$

$$= \tfrac{9}{64}(1 + {\cdot}03125 + {\cdot}00122)$$

$$= 0{\cdot}145.$$

$$\text{Change in horizontal radius} = \frac{-2 \times 2240 \times 27}{2 \times 30 \times 10^6 \times {\cdot}2485}\left[\frac{2 \times 9}{\pi({\cdot}145 + 9)} - \tfrac{1}{2}\right]$$

$$= -{\cdot}0081[{\cdot}626 - {\cdot}5]$$

$$= -{\cdot}00102 \text{ in. increase in radius.}$$

Therefore increase in diameter $= {\cdot}00204$ in.

$$\text{Change in vertical radius} = \frac{W\rho_0^3}{2EI}\left[\frac{2\rho_0^2}{\pi(h^2 + \rho_0^2)} - \tfrac{1}{4}\pi\right]$$

$$= \frac{2 \times 2240 \times 27}{2 \times 30 \times 10^6 \times {\cdot}2485}\left[\frac{2 \times 9}{\pi({\cdot}145 + 9)} - \tfrac{1}{4}\pi\right]$$

$$= -{\cdot}0081[{\cdot}626 - {\cdot}785]$$

$$= +{\cdot}00129 \text{ in.}$$

Hence reduction in vertical diameter $= {\cdot}00258$ in.

4. Links.

The stress in a link which is composed of semicircular ends with straight sides may be deduced from the results for the stresses in rings.

Cutting the semicircular end at AB and CD, then the forces P and W and couples M_1 and M_2 must act on the portion ABCD as in the case of a ring (fig. 11.11). The moment about E is

$$M_2 = M_1 + \tfrac{1}{2}W\rho_0(1 - \cos\theta).$$

But the bending moment of a curved beam is given by equation (5), p. 257, and is

$$M_2 = E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2. \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore\ E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2 = M_1 + \tfrac{1}{2}W\rho_0(1 - \cos\theta). \quad (33)$$

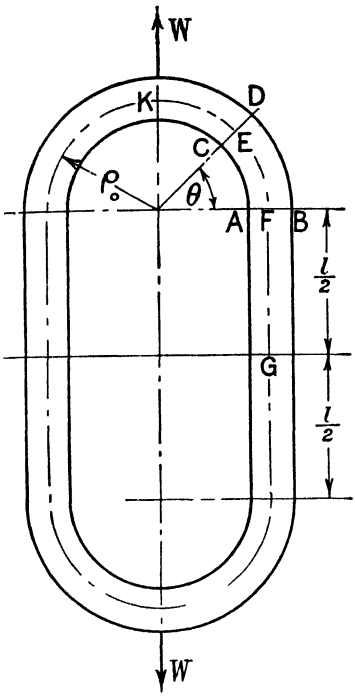


Fig 11.10

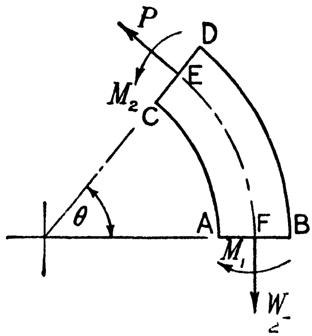


Fig 11.11

The force on a section of a curved beam is given by equation (4), p. 257, and is

$$P = EA\left[e_0 - (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}\right].$$

But the force $P = \frac{1}{2}W\cos\theta$,

$$\therefore\ EA\left[e_0 - (1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\frac{h^2}{\rho_0}\right] = \frac{1}{2}W\cos\theta. \quad . \quad (34)$$

Substituting in (34) for $E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2$ from (5) and (33), we have

$$EAe_0 - \frac{M_2}{\rho_0} = \frac{1}{2}W\cos\theta.$$

$$\therefore\ EAe_0 - \frac{M_1 + \frac{1}{2}W\rho_0(1 - \cos\theta)}{\rho_0} = \frac{1}{2}W\cos\theta.$$

$$\therefore\ EAe_0 - \frac{M_1}{\rho_0} - \frac{1}{2}W + \frac{1}{2}W\cos\theta = \frac{1}{2}\cos W\theta.$$

$$\therefore\ EAe_0 = \frac{M_1}{\rho_0} + \frac{1}{2}W. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Hence e_0 is a constant.

The change of slope of the curve of flexure $= \dfrac{M\,ds}{EI}$

$$= \frac{M}{EI}\rho_0\,d\theta,$$

where $d\theta$ is the angular change. Hence from (33),

$$E(1+e_0)\left(\frac{1}{\rho_1}-\frac{1}{\rho_0}\right)Ah^2\rho_0\frac{d\theta}{EI} = \left\{M_1 + \tfrac{1}{2}W\rho_0(1-\cos\theta)\right\}\rho_0\frac{d\theta}{EI}.$$

For the complete quadrant,

$$E(1+e_0)\left(\frac{1}{\rho_1}-\frac{1}{\rho_0}\right)Ah^2\rho_0\int_0^{\frac{1}{2}\pi}d\theta$$

$$= M_1\rho_0\int_0^{\frac{1}{2}\pi}d\theta + \tfrac{1}{2}W\rho_0^2\int_0^{\frac{1}{2}\pi}(1-\cos\theta)\,d\theta. \quad (36)$$

After straining, the length of the axis EF is

$$\rho_1\theta_1 = \int_0^{\theta_1}(1+e_0)\rho_0\,d\theta.$$

The length for the quadrant $= \int_0^{\frac{1}{2}\pi}(1+e_0)\rho_0\,d\theta$. But the angle between the tangents to the curve of flexure at F and K is not $\frac{1}{2}\pi$, owing to the bending in the straight length G to F. The slope of the tangent at F will be $\dfrac{M_1 \cdot \frac{1}{2}l}{2EI}$, where $\frac{1}{2}l$ is the distance GF and the angle

$$\theta_1 = \int_0^{\frac{1}{2}\pi}(1+e_0)\frac{\rho_0}{\rho_1}\,d\theta.$$

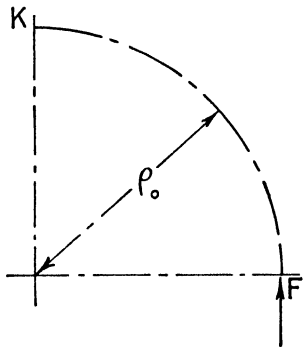


Fig. 11.12

Hence $$\int_0^{\frac{1}{2}\pi}(1+e_0)\frac{\rho_0}{\rho_1}\,d\theta = \tfrac{1}{2}\pi - \frac{\frac{1}{4}M_1 l}{EI}. \quad . \quad . \quad . \quad (37)$$

Now equation (36) can be expanded to

$$E(1+e_0)Ah^2\frac{\rho_0}{\rho_1}\int_0^{\frac{1}{2}\pi}d\theta - E(1+e_0)Ah^2\int_0^{\frac{1}{2}\pi}d\theta$$
$$= M_1\rho_0 \cdot \tfrac{1}{2}\pi + \tfrac{1}{2}W\rho_0^2 \cdot \tfrac{1}{2}\pi - \tfrac{1}{2}W\rho_0^2.$$

Substituting for the integral from (37),

$$EAh^2 \cdot \tfrac{1}{2}\pi - \frac{EAh^2M_1l}{4EI} - EAh^2(1+e_0)\tfrac{1}{2}\pi = \tfrac{1}{2}\pi M_1\rho_0 + \tfrac{1}{4}\pi W\rho_0^2 - \tfrac{1}{2}W\rho_0^2,$$

$$\therefore M_1\left(\tfrac{1}{2}\pi\rho_0 + \frac{EAh^2l}{4EI}\right) = \tfrac{1}{2}W\rho_0^2(1-\tfrac{1}{2}\pi) - \tfrac{1}{2}\pi EAh^2e_0.$$

But $EAe_0 = \dfrac{M_1}{\rho_0} + \tfrac{1}{2}W$ from (35),

$$\therefore M_1\left(\tfrac{1}{2}\pi\rho_0 + \frac{EAh^2l}{4EI}\right) = \tfrac{1}{2}W\rho_0^2(1-\tfrac{1}{2}\pi) - \tfrac{1}{2}\pi h^2\left(\frac{M_1}{\rho_0} + \tfrac{1}{2}W\right).$$

$$\therefore M_1\left(\tfrac{1}{2}\pi\rho_0 + \frac{EAh^2l}{4EI} + \tfrac{1}{2}\pi\frac{h^2}{\rho_0}\right) = \tfrac{1}{2}W\rho_0^2(1-\tfrac{1}{2}\pi) - \tfrac{1}{4}\pi Wh^2.$$

$$\therefore M_1 = \frac{\tfrac{1}{2}W\rho_0^2(1-\tfrac{1}{2}\pi) - \tfrac{1}{4}\pi Wh^2}{\tfrac{1}{2}\pi\rho_0 + \dfrac{Ah^2l}{4I} + \dfrac{\tfrac{1}{2}\pi h^2}{\rho_0}}$$

$$= \frac{W\left(\dfrac{\rho_0^2}{\pi} - \tfrac{1}{2}\rho_0^2 - \tfrac{1}{2}h^2\right)}{\rho_0 + \dfrac{h^2l}{2\pi k^2} + \dfrac{h^2}{\rho_0}}, \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

where $k^2 = \dfrac{I}{A}$, i.e. k is the radius of gyration.

But $\quad M_2 = M_1 + \tfrac{1}{2}W\rho_0(1-\cos\theta)$

$$= \frac{W\left(\dfrac{\rho_0^2}{\pi} - \tfrac{1}{2}\rho_0^2 - \tfrac{1}{2}h^2\right)}{\rho_0 + \dfrac{h^2l}{2\pi k^2} + \dfrac{h^2}{\rho_0}} + \tfrac{1}{2}W\rho_0(1-\cos\theta). \qquad (39)$$

The stress is given by equation (1), p. 255, and is

$$f = eE = E\left[\frac{(1+e_0)\left(\dfrac{z}{\rho_1} - \dfrac{z}{\rho_0}\right)}{1+\dfrac{z}{\rho_0}} + e_0\right]. \quad . \quad . \quad . \quad (1)$$

The bending moment is given by equation (5), p. 257, and is

$$E(1 + e_0)\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)Ah^2 = M_2; \quad . \quad . \quad . \quad . \quad (5)$$

and the strain e_0 by equation (35) is

$$e_0 = \frac{1}{EA}\left(\frac{M_1}{\rho_0} + \tfrac{1}{2}W\right). \quad . \quad . \quad . \quad . \quad . \quad (35)$$

By using equations (1), (5), (33) and (35), the value of the stress may be obtained.

Example 5.—A chain link has semicircular ends of 3 in. radius and is made of steel of 1 in. diameter. If it is subjected to a diametrical pull of 2000 lb., and the length of the straight sides is 4 in. long, find the maximum tensile and compressive stresses in the steel. $E = 30 \times 10^6$ lb. per sq. in.

Apply equation (38), namely,

$$M_1 = \frac{W\left(\frac{\rho_0^2}{\pi} - \frac{1}{2}\rho_0^2 - \frac{1}{2}h^2\right)}{\rho_0 + \frac{h^2 l}{2\pi k^2} + \frac{h^2}{\rho_0}},$$

where $W = 2000$ lb., $A = \frac{1}{4}\pi$ sq. in., $\rho_0 = 3$ in.

$$h^2 = \frac{D^2}{16}\left\{1 + \frac{1}{2}\left(\frac{D}{2\rho_0}\right)^2 + \frac{5}{16}\left(\frac{D}{2\rho_0}\right)^4\right\},$$

where $D = 1$ in.,

$$\therefore\ h^2 = \tfrac{1}{16}\left\{1 + \tfrac{1}{2}(\tfrac{1}{6})^2 + \tfrac{5}{16}(\tfrac{1}{6})^4\right\}$$

$$= \tfrac{1}{16}\{1 + {\cdot}01389 + {\cdot}00024\}$$

$$= 0{\cdot}06338 \text{ in.}^2$$

$$k^2 = \frac{I}{A} = \frac{\frac{1}{64}\pi D^4}{\frac{1}{4}\pi D^2} = \tfrac{1}{16} \text{ in.}^2 \text{ units},$$

$$l = 4 \text{ in.}$$

$$\therefore\ M_1 = \frac{2000\left(\frac{3^2}{\pi} - \frac{1}{2} \times 3^2 - \frac{1}{2} \times {\cdot}06338\right)}{3 + \frac{{\cdot}06338 \times 4}{2\pi(\frac{1}{16})} + \frac{{\cdot}06338}{3}}$$

$$= \frac{2000(2{\cdot}865 - 4{\cdot}5 - {\cdot}03169)}{3 + 0{\cdot}645 + {\cdot}02113}$$

$$= -910 \text{ lb.-in.}$$

From equation (35), $e_0 = \frac{1}{EA}\left(\frac{M_1}{\rho_0} + \frac{1}{2}W\right)$,

$$\therefore\ e_0 = \frac{1 \times 4}{30 \times 10^6 \times \pi}\left(\frac{-910}{3} + \frac{2000}{2}\right)$$

$$= \cdot 0000296.$$

Then $$M_2 = M_1 + \tfrac{1}{2}W\rho_0(1 - \cos\theta)$$

$$= -910 + \tfrac{1}{2} \times 2000 \times 3(1 - \cos\theta).$$

When $\theta = 0$, $\cos\theta = 1$, and $M_2 = M_1 = -910$ lb.-in.

When $\theta = \frac{1}{2}\pi$, $\cos\theta = 0$, and $M_2 = -910 + 3000 = 2090$ lb.-in.

Equation (1) is $$f = E\left[\frac{(1 + e_0)\left(\frac{z}{\rho_1} - \frac{z}{\rho_0}\right)}{1 + \frac{z}{\rho_0}} + e_0\right].$$

Substituting from (5),

$$f = \frac{zM_2}{Ah^2\left(1 + \frac{z}{\rho_0}\right)} + Ee_0$$

$$= \frac{2090z}{\frac{1}{4}\pi \times \cdot 06338(1 + \frac{1}{3}z)} + 30 \times 10^6 \times 0{\cdot}0000296$$

(for top of link)

$$= \frac{42{,}000z}{1 + \frac{1}{3}z} + 888.$$

At outer edge of link, $z = +\frac{1}{2}$ in.,

$$\therefore\ f = \frac{42{,}000 \times \frac{1}{2}}{1 + \frac{1}{3} \times \frac{1}{2}} + 888$$

$$= 18{,}000 + 888$$

$$= 18{,}888 \text{ lb. per sq. in. tension.}$$

At inner edge of link, $z = -\frac{1}{2}$ in.,

$$\therefore\ f = \frac{42{,}000(-\frac{1}{2})}{1 - \frac{1}{6}} + 888$$

$$= -25{,}200 + 888$$

$$= -24{,}312 \text{ lb. per sq. in. compression.}$$

When $\theta = 0$, $M_2 = M_1 = -910$ lb.-in., then

$$f = \frac{-910z}{\frac{1}{4}\pi \times \cdot 06338(1 + \frac{1}{3}z)} + 30 \times 10^6 \times \cdot 0000296 + \frac{W}{2A}$$

$$\left(\frac{W}{2A} = \text{direct stress}\right)$$

$$= -\frac{18{,}300z}{1 + \frac{1}{3}z} + 888 + \frac{2000 \times 2}{\pi}$$

$$= -\frac{18{,}300z}{1 + \frac{1}{3}z} + 2161.$$

At outer edge, $z = +\frac{1}{2}$ in.,

$$\therefore f = -\frac{18{,}300 \times \frac{1}{2}}{1 + \frac{1}{3} \times \frac{1}{2}} + 2161$$

$$= -7840 + 2161$$

$$= -5680 \text{ lb. per sq. in. compression.}$$

At inner edge, $z = -\frac{1}{2}$ in.,

$$\therefore f = -\frac{18{,}300(-\frac{1}{2})}{1 - \frac{1}{6}} + 2161$$

$$= +10{,}980 + 2161$$

$$= 13{,}140 \text{ lb. per sq. in. tension.}$$

EXAMPLES

1. A bar of rectangular cross-section, breadth b and depth d, has initial radius of curvature R measured to the axis through the centroid of the section. The plane of curvature is parallel to two of the faces of the rectangle. Pure couples of magnitude M are applied to the end of the bar, tending to straighten it. Assuming that plane sections remain plane, show that the neutral axis will be a distance h from the axis through the centroid of the section such that

$$\frac{d}{R - h} = \log_e \frac{2R + d}{2R - d}.$$

Show also that the stress in the bar at a distance y from the neutral axis is

$$\frac{M}{bdh} \cdot \frac{y}{R - h - y}. \qquad \text{(London Univ.)}$$

2. The link of a chain is made from iron bar of square cross-section with sides 1 in. wide. The centre-line at each end is a semicircle of radius 2 in.,

and the straight portion at each side is 3 in. in length. If the link is subjected to forces of 1 ton at the end of the link, find the maximum bending moment in the link, and the maximum tensile and compressive stresses in the steel. $E = 30 \times 10^6$ lb. per sq. in.

[1618 lb.-in.; 7129 lb. per sq. in. tension; 9611 lb. per sq. in. compression.]

3. The cross-section of a crane hook is a trapezium of inner width 2 in., outer width 1 in., and depth 3 in. A load of 3 tons acts through the centre of curvature of the link and at 3 in. from the centroid of the section on which the stress is a maximum. Find the maximum stresses in the material.

[1·125 tons per sq. in.; 2·286 tons per sq. in.]

4. A ring of 3 in. mean diameter is made from a round bar of 1 in. diameter. When a diametrical pull of 5 tons is made, find the maximum stress in the steel. [31·6 tons per sq. in.]

5. The maximum stress induced in a steel tyre is 10 tons per sq. in. when a diametrical compressive load is supported. If the tyre is 4 ft. mean diameter, 4 in. wide and $1\frac{1}{2}$ in. thick, find the maximum decrease in the diameter. $E = 30 \times 10^6$ lb. per sq. in. [0·26 in.]

CHAPTER XII

Stresses due to Rotation

Exact solutions of this problem are generally impossible, but the stresses set up in the material may be found, subject to certain assumptions. It is assumed that the axis of rotation is the direction of one of the principal stresses, so that the radial and circumferential or hoop stresses are each principal stresses, and the principal stress along the axis of rotation is zero.

1. Rotating Disc.

The thickness of the disc is uniform and it is thin, so that no variation in the stress through the thickness occurs.

Let u = radial displacement at radius r.

$$\text{The circumferential strain} = \frac{2\pi(r+u) - 2\pi r}{2\pi r}$$

$$= \frac{u}{r}.$$

The radial width of an element of radius r and thickness dr after strain

$$= r + dr + u + du - (r + u)$$

$$= dr + du,$$

$$\text{and the radial strain} = \frac{dr + du - dr}{dr}$$

$$= \frac{du}{dr}.$$

Let p_x = radial stress at radius r,

p_y = circumferential stress at radius r,

E = Young's modulus,

$1/m$ = Poisson's ratio,

then $$\frac{du}{dr} = \frac{p_x}{E} - \frac{p_y}{mE} \quad \text{(see p. 182)}$$

$$= \frac{1}{E}\left(p_x - \frac{p_y}{m}\right), \quad \ldots \ldots \quad (1)$$

and $$\frac{u}{r} = \frac{1}{E}\left(p_y - \frac{p_x}{m}\right). \quad \ldots \ldots \quad (2)$$

Before a solution can be effected, another equation is necessary. This equation may be obtained by considering the forces on a small

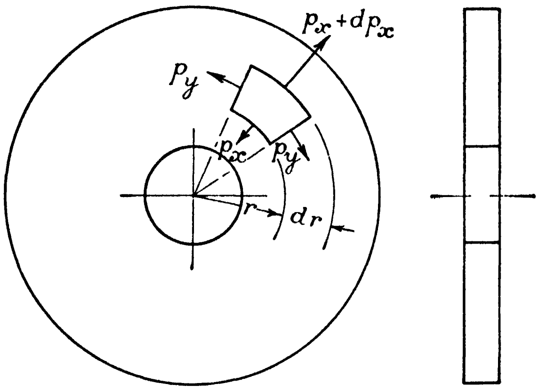


Fig. 12.1

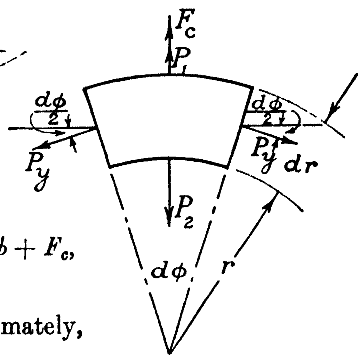
prism situated in the disc at radius r. The algebraic sum of the forces on the prism must be zero for equilibrium. Taking compressive forces as negative and resolving the forces on the prism outwards, i.e. along the direction F_c, if

F_c = centrifugal force,

$P_1 = (p_x + dp_x)(r + dr)\, d\phi$,

$P_2 = p_x r\, d\phi$,

$P_y = p_y\, dr$,



we have

$$-P_2 + P_1 - P_y \sin \tfrac{1}{2}d\phi - P_y \sin \tfrac{1}{2}d\phi + F_c,$$

or

$$-P_2 + P_1 - 2P_y \tfrac{1}{2}d\phi + F_c \text{ approximately,}$$

is zero since $\sin \frac{1}{2}d\phi \simeq \frac{1}{2}d\phi$ radians;

Fig. 12.2

then $$-p_x r\,d\phi + (p_x + dp_x)(r + dr)\,d\phi - 2p_y\,dr\,.\,\tfrac{1}{2}d\phi + \frac{w}{g}(r + \tfrac{1}{2}dr)\,dr\,d\phi\omega^2(r + \tfrac{1}{2}dr) = 0.$$

A term for the thickness is neglected, since it would appear in each term, and $w(r + \frac{1}{2}dr)\,dr\,d\phi\omega^2(r + \frac{1}{2}dr)$ is the centrifugal force on the prism, where

w = weight of one cubic in. of material,

ω = angular velocity in radians per second.

Hence $$p_y\,dr - p_x\,dr - r\,dp_x = \frac{w}{g}\,\omega^2 r^2\,dr,$$

neglecting squares and products of small quantities dp and dr.

$$\therefore\ p_y = p_x + r\frac{dp_x}{dr} + \frac{w}{g}\,\omega^2 r^2, \quad . \quad . \quad . \quad . \quad (3)$$

i.e. $$p_y = \frac{w}{g}\,\omega^2 r^2 + \frac{d}{dr}(rp_x).$$

$$\therefore\ \frac{d}{dr}(rp_x) = p_y - \frac{w}{g}\,\omega^2 r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By differentiating equation (2) with respect to r,

$$\frac{du}{dr} = \frac{d}{dr}\left\{\frac{r}{E}\left(p_y - \frac{p_x}{m}\right)\right\}$$

$$= \frac{d}{dr}\left(\frac{rp_y}{E}\right) - \frac{d}{dr}\left(\frac{rp_x}{mE}\right).$$

Substituting for $\frac{d}{dr}(rp_x)$ from (4),

$$\frac{du}{dr} = \frac{d}{dr}\left(\frac{rp_y}{E}\right) - \frac{p_y}{mE} + \frac{w\omega^2 r^2}{mgE}.$$

But from (1), $$\frac{du}{dr} = \frac{1}{E}\left(p_x - \frac{p_y}{m}\right),$$

$$\therefore\ \frac{1}{E}\left(p_x - \frac{p_y}{m}\right) = \frac{d}{dr}\left(\frac{rp_y}{E}\right) - \frac{p_y}{mE} + \frac{w\omega^2 r^2}{mgE}.$$

$$\therefore\ p_x - \frac{p_y}{m} = \frac{d}{dr}rp_y - \frac{p_y}{m} + \frac{w\omega^2 r^2}{mg}.$$

$$\therefore\ p_x = \frac{d}{dr}(rp_y) + \frac{w\omega^2 r^2}{mg}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Multiplying throughout by r and differentiating with respect to r,

$$\frac{d}{dr}(rp_x) = \frac{d}{dr}\left\{r\frac{d}{dr}(rp_y) + \frac{w\omega^2 r^3}{gm}\right\}.$$

Writing y for rp_y and substituting for $\frac{d}{dr}(rp_x)$ from (4),

$$p_y - \frac{w}{g}\omega^2 r^2 = \frac{d}{dr}\left(r\frac{dy}{dr}\right) + \frac{3w\omega^2 r^2}{gm},$$

or

$$\frac{y}{r} - \frac{w}{g}\omega^2 r^2 = r\frac{d^2y}{dr^2} + \frac{dy}{dr} + \frac{3w\omega^2 r^2}{gm}.$$

$$\therefore\ r\frac{d^2y}{dr^2} + \frac{dy}{dr} - \frac{y}{r} = -\left(1 + \frac{3}{m}\right)\frac{w}{g}\omega^2 r^2.$$

Multiplying throughout by r, a homogeneous linear equation is obtained:

$$r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} - y = -\left(1 + \frac{3}{m}\right)\frac{w}{g}\omega^2 r^3.$$

To obtain the complementary function, substitute r^n for y and equate the left-hand side to zero, i.e.

$$r^2 n(n-1)r^{n-2} + rnr^{n-1} - r^n = 0.$$

$$\therefore\ r^n(n^2 - n + n - 1) = 0,$$

$$\therefore\ n^2 - 1 = 0,$$

$$\therefore\ n = \pm 1.$$

$$\therefore\ \text{Complementary function} = Ar + \frac{B}{r},$$

where A and B are constants.

For the particular integral, substitute Hr^n for y,

$$Hr^2 n(n-1)r^{n-2} + Hrnr^{n-1} - Hr^n = -\left(1 + \frac{3}{m}\right)\frac{w}{g}\omega^2 r^3,$$

i.e.

$$Hr^n(n^2 - 1) = -\left(1 + \frac{3}{m}\right)\frac{w}{g}\omega^2 r^3.$$

$$\therefore\ r^n = r^3,$$

$$\therefore\ n = 3.$$

$$Hr^3 \times 8 = -\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^3,$$

$$\therefore\ 8H = -\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2,$$

$$\therefore\ H = -\tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2.$$

$$\therefore\ y = -\tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^3.$$

The complete solution is

$$y = Ar + \frac{B}{r} - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^3.$$

But $\frac{y}{r} = p_y$,

$$\therefore\ p_y = A + \frac{B}{r^2} - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^2.$$

From (5), $$p_x = \frac{d}{dr}(rp_y) + \frac{w\omega^2 r^2}{mg}$$

$$= \frac{d}{dr}\left\{Ar + \frac{B}{r} - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^3\right\} + \frac{w\omega^2 r^2}{mg}$$

$$= A - \frac{B}{r^2} - \tfrac{3}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^2 + \frac{w\omega^2 r^2}{mg}$$

$$= A - \frac{B}{r^2} - \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 r^2.$$

The values of A and B must be found to satisfy the given conditions. Generally two cases are needed: (1) solid disc, (2) hollow disc.

Case (1).—If B is not zero, p_y and p_x approach infinity when r is very small; but the stresses are finite for all values of r, therefore B must be zero.

The remaining condition is $p_x = 0$ when $r = R =$ outside radius,

$$\therefore\ 0 = A - \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R^2,$$

i.e. $$A = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R^2.$$

Hence $$p_y = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R^2 - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^2, \quad . \quad . \quad (6)$$

and $$p_x = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R^2 - \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 r^2$$

$$= \frac{w\omega^2}{8g}\left(3 + \frac{1}{m}\right)(R^2 - r^2), \qquad (7)$$

and $$u = \frac{r}{E}\left(p_y - \frac{p_x}{m}\right). \qquad (8)$$

The maximum values of p_x and p_y occur at the centre, when $r = 0$, then

$$p_y = p_x = \frac{w}{8g}\,\omega^2\left(3 + \frac{1}{m}\right)R^2. \qquad (9)$$

Case (2).—Let R_1 = internal radius,

and R_2 = external radius,

then $p_x = 0$ when $r = R_1$ and $r = R_2$.

Substituting these values in the equation for p_x,

$$p_x = 0 = A - \frac{B}{R_1^2} - \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R_1^2$$

$$= 0 = A - \frac{B}{R_2^2} - \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R_2^2.$$

By subtraction,

$$B\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right) = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2(R_1^2 - R_2^2),$$

or $$B\left(\frac{R_1^2 - R_2^2}{R_1^2 R_2^2}\right) = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2(R_1^2 - R_2^2).$$

$$\therefore\ B = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R_1^2 R_2^2,$$

and $$A = \frac{B}{R_1^2} + \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R_1^2$$

$$= \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R_2^2 + \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R_1^2$$

$$= \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2(R_2^2 + R_1^2).$$

$$\therefore\ p_y = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2\left(R_2^2 + R_1^2 + \frac{R_1^2 R_2^2}{r^2}\right) - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2 r^2 \qquad (10)$$

and $$p_x = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2\left(R_2^2 + R_1^2 - \frac{R_1^2R_2^2}{r^2} - r^2\right). \qquad (11)$$

The greatest value of p_y occurs when $r = R_1$, i.e. at the inner radius, whilst p_x is a maximum when $r = \sqrt{R_1R_2}$.

When $r = R_1$,

$$p_y = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2(2R_2^2 + R_1^2) - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2R_1^2$$

$$= \tfrac{1}{8}\frac{w}{g}\,\omega^2\left[\left(3 + \frac{1}{m}\right)(2R_2^2 + R_1^2) - \left(1 + \frac{3}{m}\right)R_1^2\right]$$

$$= \tfrac{1}{8}\frac{w\omega^2}{gm}\left[(3m + 1)(2R_2^2 + R_1^2) - (m + 3)R_1^2\right]$$

$$= \frac{(3m + 1)w\omega^2}{4gm}\left[R_2^2 + \frac{(m - 1)}{3m + 1}R_1^2\right]. \qquad (12)$$

It should be noted that, if R_1^2 is negligible compared with R_2^2, the maximum circumferential stress

$$p_y = \frac{(3m + 1)w\omega^2}{4gm}R_2^2,$$

which is twice the value when the disc is solid [see equation (9)]. Hence a small hole in the disc doubles the maximum circumferential stress.

Example 1.—A solid thin disc, 12 in. radius, is revolved at 8000 revolutions per minute. If the density of the material is 480 lb. per cubic foot, and Poisson's ratio is 0·3, find the circumferential and radial stresses in the disc and the radial displacements. Plot graphs of the values at their respective radii. $E = 30 \times 10^6$ lb. per sq. in.

$$p_y = \tfrac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2R^2 - \tfrac{1}{8}\left(1 + \frac{3}{m}\right)\frac{w}{g}\,\omega^2r^2 \quad \text{[equation (6)]}$$

$$= \frac{w\omega^2}{8gm}\left[(3m + 1)R^2 - (m + 3)r^2\right],$$

where $w = 480$ lb. per c. ft $= \frac{480}{1728}$ lb. per c. in.,

$g = 32{\cdot}2$ ft. per sec.^{2} $= 32{\cdot}2 \times 12$ in. per sec.^{2},

$\omega = \dfrac{2\pi \times 8000}{60}$ radians per sec.,

$1/m = 0{\cdot}3$, $\therefore\ m = 10/3$.

$$p_y = \frac{480 \times 4\pi^2 \times 8000^2 \times 3}{1728 \times 60^2 \times 8 \times 32{\cdot}2 \times 12 \times 10}\left[\left(\tfrac{30}{3} + 1\right)12^2 - \left(\tfrac{10}{3} + 3\right)r^2\right]$$

$$= 18{\cdot}92\,[1584 - 6{\cdot}33r^2].$$

$$p_x = \frac{w\omega^2}{8g}\left(3 + \frac{1}{m}\right)(R^2 - r^2) \qquad \text{[equation (7)]}$$

$$= \frac{480 \times 4\pi^2 \times 8000^2}{1728 \times 60^2 \times 8 \times 32{\cdot}2 \times 12}(3 + \tfrac{3}{10})(12^2 - r^2)$$

$$= 208(12^2 - r^2).$$

$$u = \frac{r}{E}\left(p_y - \frac{p_x}{m}\right). \qquad \text{[equation (8)]}$$

Radius, in.	p_y lb. per sq. in.	p_x lb. per sq. in.	u, in.*
0	30,000	30,000	0
2	29,500	29 100	0·00138
4	28,100	26,600	0·00268
6	25,700	22,450	0·0038
8	22,300	16,600	0·0046
10	18,000	9,150	0·0051
12	12,700	0	0·0051

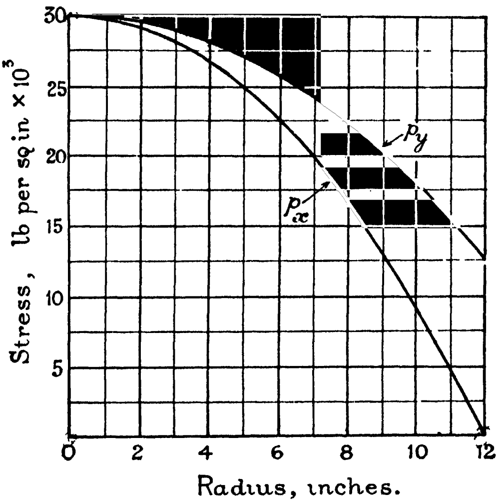


Fig 12.3

Applying the various failure theories:

(i) Maximum-principal-stress theory.— The limiting stress is the limit of proportionality stress (f). Then $p_y = f$.

(ii) Maximum-shear-stress theory.—The maximum shear stress equals one half the maximum difference between the principal stresses. The principal stresses are p_x, p_y and p_z. p_z equals zero, hence the

* See diagram on p. 288.

maximum shear stress, which is $\frac{1}{2}f$ at the limit of proportionality, is

$$\tfrac{1}{2}f = \tfrac{1}{2}(p_x - p_z),$$

$$\therefore f = p_x$$

if p_x and p_y are of the same type of stress, and p_x is greater than p_y.

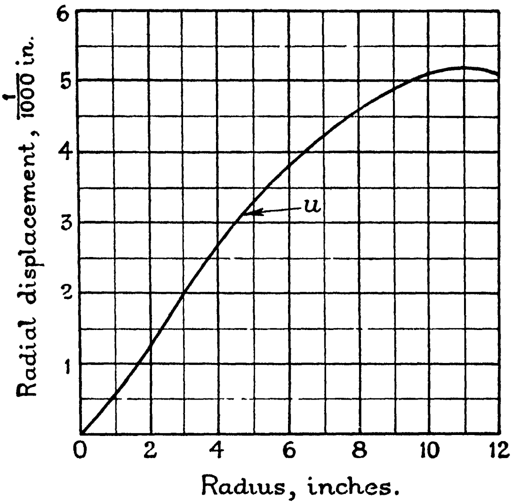


Fig. 12.4

(iii) Maximum-strain theory.—The strain must not be greater than the strain produced by the stress at the limit of proportionality, i.e. f/E.

$$\text{The circumferential strain} = \frac{1}{E}\left(p_y - \frac{p_x}{m}\right).$$

$$\therefore f = \left(p_y - \frac{p_x}{m}\right).$$

(iv) Maximum-strain-energy theory.—The work done per unit volume by the limit of proportionality stress is $\frac{1}{2}f^2/E$.

The stress-strain energy per unit volume for a two-dimensional case in which the stresses are of the same type is (p. 187)

$$\frac{1}{2E}\left(p_x^2 + p_y^2 - \frac{2}{m}p_xp_y\right).$$

$$\therefore \tfrac{1}{2}\frac{f^2}{E} = \frac{1}{2E}\left(p_x^2 + p_y^2 - \frac{2}{m}p_xp_y\right).$$

$$\therefore f^2 = p_x^2 + p_y^2 - \frac{2}{m}p_xp_y.$$

The maximum safe speed for a solid disc may be deduced in accordance with each of the above theories.

First Case.— $f = p_y = \frac{1}{8}\left(3 + \frac{1}{m}\right)\frac{w}{g}\,\omega^2 R^2.$ [equation (9)]

$$\therefore\ \omega^2 = \frac{8gf}{\left(3 + \frac{1}{m}\right)wR^2}.$$

Second Case.—The maximum difference in p_r and p_y will occur when $r = 0$.

$$f = p_x - \left(3 + \frac{1}{m}\right)\frac{w\omega^2 R^2}{8g}. \qquad \text{[equation (7)]}$$

$$\therefore\ \omega^2 = \frac{8gf}{\left(3 + \frac{1}{m}\right)wR^2}.$$

Third Case.—The maximum difference between p_y and p_x/m occurs when $r = 0$. From equations (6) and (7).

$$f = \left(3 + \frac{1}{m}\right)\frac{w\omega^2 R^2}{8g} - \left(3 + \frac{1}{m}\right)\frac{w\omega^2 R^2}{8gm}.$$

$$\therefore\ \omega^2 = \frac{8gf}{\left(3 + \frac{1}{m}\right)\left(1 - \frac{1}{m}\right)wR^2}.$$

Fourth Case.—The maximum condition occurs when $r = 0$, then $p_x = p_y$.

$$f^2 = 2p_x^2 - \frac{2}{m}p_x^2$$

$$= p_x^2\left(2 - \frac{2}{m}\right)$$

$$= \left[\frac{w\omega^2}{8g}\left(3 + \frac{1}{m}\right)R^2\right]^2\left(2 - \frac{2}{m}\right).$$

$$\therefore\ \omega^2 = \frac{8gf}{\left(3 + \frac{1}{m}\right)wR^2}\cdot\frac{1}{\sqrt{\left(2 - \frac{2}{m}\right)}}$$

$$= \frac{8gf}{\left(3 + \frac{1}{m}\right)wR^2}\sqrt{\frac{m}{2(m-1)}}.$$

Example 2.—Using the four failure theories, find the maximum speed for a 20 in. diameter disc if the limit of proportionality is 15 tons per sq. in., the density of the material 480 lb. per c. ft. and Poisson's ratio 0·3.

Case 1.—$\omega^2 = \dfrac{8 \times 32{\cdot}2 \times 12 \times 15 \times 2240 \times 1728}{(3 + 0{\cdot}3)480 \times 10^2}$.

$\therefore\ \omega = 1065$ radians per sec.

$= 10{,}200$ revolutions per minute.

Case 2.—This is the same as Case 1.

Case 3.—$\omega^2 = \dfrac{8 \times 32{\cdot}2 \times 12 \times 15 \times 2240 \times 1728}{(3 + {\cdot}3)(1 - {\cdot}3)480 \times 10^2}$.

$\therefore\ \omega = 1270$ radians per sec.

$= 12{,}100$ revolutions per minute.

Case 4.—$\omega^2 = \dfrac{8 \times 32{\cdot}2 \times 12 \times 15 \times 2240 \times 1728}{(3 + {\cdot}3)480 \times 10^2}\sqrt{\dfrac{\frac{10}{3}}{2(\frac{10}{3} - 1)}}$.

$\therefore\ \omega = 980$ radians per second

$= 9350$ revolutions per minute.

2. Rotating Cylinder.

In the work on flat discs we dealt with a two-dimensional stress, but, when the width of the disc is increased, the axial stress must be considered. In order that plane cross-sections may be considered to remain plane when under stress, it must be assumed that the axial stress is constant. If the cylinder is long this assumption must be approximately correct for all sections except those near the ends. The three stresses, i.e. radial, circumferential, and axial, will all be principal stresses.

As for the thin disc, the radial strain will be

$$\frac{du}{dr} = \frac{p_x}{E} - \frac{p_y}{mE} - \frac{p_z}{mE}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the circumferential strain

$$\frac{u}{r} = \frac{p_y}{E} - \frac{p_x}{mE} - \frac{p_z}{mE}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and the axial strain

$$e_z = \frac{p_z}{E} - \frac{p_y}{mE} - \frac{p_x}{mE}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where p_z is the axial stress.

The equation of the forces acting on a prism of material will be the same as those acting in the disc, namely,

$$p_y = p_x + r\frac{dp_x}{dr} + \frac{w}{g}\omega^2 r^2, \qquad \text{[equation (3)]}$$

$$\frac{d}{dr}(rp_x) = p_y - \frac{w}{g}\omega^2 r^2. \qquad \text{[equation (4)]}$$

Differentiating equation (14) with respect to r,

$$E\frac{du}{dr} = p_y - \frac{1}{m}p_x - \frac{1}{m}p_z + r\frac{dp_y}{dr} - \frac{r}{m}\frac{dp_x}{dr} - \frac{r}{m}\frac{dp_z}{dr}.$$

But
$$E\frac{du}{dr} = p_x - \frac{p_y}{m} - \frac{p_z}{m}, \qquad \text{from (13),}$$

$$\therefore\ p_x - \frac{p_y}{m} - \frac{p_z}{m} = p_y - \frac{p_x}{m} - \frac{p_z}{m} + r\frac{dp_y}{dr} - \frac{r}{m}\frac{dp_x}{dr} - \frac{r}{m}\frac{dp_z}{dr}.$$

$$\therefore\ p_x - \frac{p_y}{m} - p_y - \frac{p_x}{m} + r\frac{dp_y}{dr} - \frac{r}{m}\frac{dp_x}{dr} - \frac{r}{m}\frac{dp_z}{dr}.$$

$$\therefore\ (p_x - p_y)\left(1 + \frac{1}{m}\right) = r\frac{dp_y}{dr} - \frac{r}{m}\frac{dp_x}{dr} - \frac{r}{m}\frac{dp_z}{dr} \quad . \quad . \quad (16)$$

But the axial strain e_z is a constant, therefore equation (15) is

$$Ee_z = p_z - \frac{p_y}{m} - \frac{p_x}{m} = \text{constant.} \quad . \quad . \quad . \quad (15')$$

Hence
$$p_z = \frac{p_y}{m} + \frac{p_x}{m} + \text{constant,}$$

and
$$\frac{dp_z}{dr} = \frac{1}{m}\frac{dp_y}{dr} + \frac{1}{m}\frac{dp_x}{dr}. \quad . \quad . \quad . \quad . \quad (17)$$

Substituting for $\frac{dp_z}{dr}$ from (17) in (16),

$$(p_x - p_y)\left(1 + \frac{1}{m}\right) = \frac{rdp_y}{dr} - \frac{r}{m}\frac{dp_x}{dr} - \frac{r}{m^2}\frac{dp_y}{dr} - \frac{r}{m^2}\frac{dp_x}{dr}$$

$$= r\frac{dp_y}{dr}\left(1 - \frac{1}{m^2}\right) - \frac{r}{m}\frac{dp_x}{dr}\left(1 + \frac{1}{m}\right).$$

$$\therefore\ (p_x - p_y) = r\frac{dp_y}{dr}\left(1 - \frac{1}{m}\right) - \frac{r}{m}\frac{dp_x}{dr}. \quad . \quad . \quad . \quad (18)$$

From (3), $$p_x - p_y = -r\frac{dp_x}{dr} - \frac{w}{g}\omega^2 r^2.$$

$$\therefore\ -r\frac{dp_x}{dr} - \frac{w}{g}\omega^2 r^2 = r\frac{dp_y}{dr}\left(1 - \frac{1}{m}\right) - \frac{r}{m}\frac{dp_x}{dr}.$$

$$\therefore\ r\frac{dp_y}{dr}\left(1 - \frac{1}{m}\right) + r\frac{dp_x}{dr}\left(1 - \frac{1}{m}\right) = -\frac{w}{g}\omega^2 r^2.$$

$$\therefore\ \frac{dp_y}{dr} + \frac{dp_x}{dr} = -\frac{m}{m-1}\,\frac{w\omega^2}{g}\,r.$$

$$\therefore\ p_y + p_x = A - \frac{m}{m-1}\cdot\frac{w\omega^2 r^2}{2g}, \quad . \quad . \quad . \quad (19)$$

where A is a constant.

But from (3), $$p_y = p_x + r\frac{dp_x}{dr} + \frac{w}{g}\omega^2 r^2,$$

therefore (19) becomes

$$2p_x + r\frac{dp_x}{dr} + \frac{w}{g}\omega^2 r^2 = A - \frac{m}{m-1}\cdot\frac{w\omega^2 r^2}{2g}.$$

Multiplying throughout by r,

$$\frac{d}{dr}(p_x r^2) = Ar - \frac{3m-2}{2(m-1)}\cdot\frac{w}{g}\omega^2 r^3.$$

$$\therefore\ p_x r^2 = \tfrac{1}{2}Ar^2 - \frac{3m-2}{2(m-1)}\cdot\frac{w}{g}\,\frac{\omega^2 r^4}{4} + B,$$

where B is a constant.

$$\therefore\ p_x = \tfrac{1}{2}A + \frac{B}{r^2} - \frac{3m-2}{2(m-1)}\cdot\frac{w}{g}\,\frac{\omega^2 r^2}{4}. \quad . \quad . \quad (20)$$

Substituting for p_x from (20) in (19),

$$p_y = \tfrac{1}{2}A - \frac{B}{r^2} - \frac{m+2}{8(m-1)}\cdot\frac{w}{g}\omega^2 r^2. \quad . \quad . \quad . \quad (21)$$

In order to obtain the expression for p_z it will be necessary to find the value of e_z. If the ends are free to move, then there must be no resultant tension or thrust in an axial direction, i.e. the axial force must be zero.

The axial force on an elemental ring of material

$$= 2\pi r\,dr\,p_z.$$

Therefore the total axial force

$$=\int_0^r 2\pi r\, dr\, p_z = 0,$$

i.e.

$$\int_0^r r\, dr\, p_z = 0.$$

But

$$p_z = Ee_z + \frac{p_y}{m} + \frac{p_x}{m},$$

$$\therefore \int_0^r r\, dr \left[Ee_z + \frac{p_y}{m} + \frac{p_x}{m}\right] = 0.$$

$$\therefore \int_0^r r\, dr\left[Ee_z + \frac{1}{m}\left\{\tfrac{1}{2}A - \frac{B}{r^2} + \tfrac{1}{2}A + \frac{B}{r^2} - \left(\frac{m+2}{8(m-1)} + \frac{3m-2}{8(m-1)}\right)\frac{w\omega^2 r^2}{g}\right\}\right] = 0.$$

$$\therefore \int_0^r r\, dr\left[Ee_z + \frac{1}{m}\left\{A - \frac{m}{2(m-1)}\cdot\frac{w\omega^2 r^2}{g}\right\}\right] = 0. \qquad (22)$$

Solution of this equation for definite conditions will give the value of e_z. Since $p_z = Ee_z + p_y/m + p_x/m$, the value of p_z can be obtained.

The values of A and B depend upon the given conditions. Generally the cases of solid and hollow cylinders are solved.

Case 1. *Solid cylinder.*—If B is not zero, p_y and p_x approach infinity when r is very small. As this is impossible, the value of B must be zero. The remaining condition is $p_x = 0$ when $r = R =$ outside radius,

$$\therefore 0 = \tfrac{1}{2}A - \frac{3m-2}{2(m-1)}\cdot\frac{w\omega^2 R^2}{4g}.$$

$$\therefore A = \frac{3m-2}{(m-1)}\cdot\frac{w\omega^2 R^2}{4g}.$$

$$\therefore p_x = \frac{3m-2}{8(m-1)}\cdot\frac{w\omega^2 R^2}{g} - \frac{3m-2}{8(m-1)}\cdot\frac{w\omega^2 r^2}{g}$$

$$= \frac{3m-2}{8(m-1)}\cdot\frac{w\omega^2}{g}(R^2 - r^2). \quad \ldots\ldots \quad (23)$$

$$p_y = \frac{3m-2}{8(m-1)}\cdot\frac{w\omega^2 R^2}{g} - \frac{m+2}{8(m-1)}\cdot\frac{w\omega^2 r^2}{g}$$

$$= \frac{w\omega^2}{8(m-1)g}\left\{(3m-2)R^2 - (m+2)r^2\right\}. \quad . \quad (24)$$

The value of e_z and subsequently p_z may now be determined from

$$2\pi\int_0^R r\,dr\left[Ee_z+\frac{p_y}{m}+\frac{p_x}{m}\right]=0,$$

or

$$\int_0^R r\,dr\left[Ee_z+\{(3m-2)R^2-(m+2)r^2+(3m-2)(R^2-r^2)\}\frac{w\omega^2}{8(m-1)gm}\right]=0.$$

$$\therefore\ -Ee_z.\tfrac{1}{2}R^2$$

$$=[\tfrac{1}{2}(3m-2)R^4-\tfrac{1}{4}(m+2)R^4+(3m-2)(\tfrac{1}{2}R^4-\tfrac{1}{4}R^4)]\frac{w\omega^2}{8(m-1)gm}$$

$$=[\tfrac{3}{2}mR^4-R^4-\tfrac{1}{4}mR^4-\tfrac{1}{2}R^4+\tfrac{3}{4}mR^4-\tfrac{1}{2}R^4]\frac{w\omega^2}{8(m-1)gm}$$

$$=(2mR^4-2R^4)\frac{w\omega^2}{8(m-1)gm}$$

$$=2R^4(m-1)\frac{w\omega^2}{8(m-1)gm}$$

$$=\frac{R^4w\omega^2}{4gm}.$$

$$\therefore\ Ee_z=-\frac{R^2w\omega^2}{2gm}.$$

By substituting for Ee_z, p_x, and p_y in equation (15′),

$$p_z=-\frac{R^2w\omega^2}{2gm}+\{(3m-2)R^2-(m+2)r^2+(3m-2)(R^2-r^2)\}\frac{w^2\omega}{8(m-1)mg}$$

$$=-\frac{R^2w\omega^2}{2gm}+\{2(3m-2)R^2-4mr^2\}\frac{w\omega^2}{8(m-1)mg}. \quad . \quad . \quad (25)$$

At the centre of the cylinder

$$p_x=p_y=\frac{(3m-2)}{8(m-1)}\cdot\frac{w\omega^2R^2}{g}. \quad . \quad . \quad . \quad . \quad (26)$$

The stresses at the centre of a disc are given by equation (9),

$$p_x=p_y=\frac{w\omega^2}{8g}\left(3+\frac{1}{m}\right)R^2,$$

so that the ratio of the stresses at the centre of a disc to those of a cylinder are

$$\left(3+\frac{1}{m}\right)\text{ to }\frac{3m-2}{m-1}.$$

If the value of Poisson's ratio $1/m$ is $\frac{1}{4}$, the ratio becomes $3\frac{1}{4}$ to $3\frac{1}{3}$, so that there is little difference between the stresses in a disc and a cylinder.

When $r = 0$, p_z may be obtained from equation (25).

$$p_z = -\frac{R^2 w\omega^2}{2gm} + \{2(3m-2)R^2\}\frac{w\omega^2}{8(m-1)mg}$$

$$= \frac{w\omega^2}{2mg}\left[-R^2 + \frac{(3m-2)R^2}{2(m-1)}\right]$$

$$= \frac{w\omega^2}{4mg(m-1)} \cdot mR^2$$

$$= \frac{w\omega^2 R^2}{4(m-1)g}.$$

The values of p_x and p_y are equal (when $r = 0$), and are to the maximum value of p_z in the ratio of

$$\frac{3m-2}{8(m-1)} \cdot \frac{w\omega^2 R^2}{g} \text{ to } \frac{w\omega^2 R^2}{4(m-1)g},$$

i.e. $$(3m-2)/2 \text{ to } 1.$$

If $m = 4$, the ratio is $(12-2)/2$ to 1, i.e. 5 to 1.

Case 2. *Hollow cylinder.*—Let

$$R_1 = \text{internal radius},$$

and $$R_2 = \text{external radius},$$

then $p_x = 0$ when $r = R_1$ and $r = R_2$, i.e. from equation (20)

$$0 = \tfrac{1}{2}A + \frac{B}{R_1^2} - \frac{3m-2}{2(m-1)} \cdot \frac{w}{g}\frac{\omega^2 R_1^2}{4},$$

and $$0 = \tfrac{1}{2}A + \frac{B}{R_2^2} - \frac{3m-2}{2(m-1)} \cdot \frac{w}{g}\frac{\omega^2 R_2^2}{4}.$$

By subtraction,

$$0 = \frac{B}{R_1^2} - \frac{B}{R_2^2} - \frac{3m-2}{2(m-1)} \cdot \frac{w\omega^2}{4g}(R_1^2 - R_2^2).$$

$$\therefore\ B = \frac{3m-2}{8(m-1)} \cdot \frac{w\omega^2}{g}\,\frac{(R_1^2 - R_2^2) \,.\, R_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

$$= -\frac{3m-2}{8(m-1)} \cdot \frac{w\omega^2}{g} \,.\, R_1^2 R_2^2,$$

and $$A = \frac{3m-2}{4(m-1)}\frac{w\omega^2}{g}R_2^2 + \frac{3m-2}{4(m-1)}\cdot\frac{w}{g}\omega^2 R_1^2$$

$$= \frac{3m-2}{4(m-1)}\cdot\frac{w\omega^2}{g}(R_2^2 + R_1^2).$$

$$\therefore\ p_x = \frac{3m-2}{8(m-1)}\cdot\frac{w\omega^2}{g}\left(R_2^2 + R_1^2 - \frac{R_1^2R_2^2}{r^2} - r^2\right), \quad . \quad . \quad (27)$$

and $$p_y = \frac{3m-2}{8(m-1)}\cdot\frac{w\omega^2}{g}\left(R_2^2 + R_1^2 + \frac{R_1^2R_2^2}{r^2} - \frac{m+2}{3m-2}\cdot r^2\right). \quad (28)$$

3. Rotation Disc of Thickness given by Equation $t = cr^{-\beta}$.

There is very little difference between the stresses in a disc and a cylinder, so that the axial stress may be considered zero in this case.

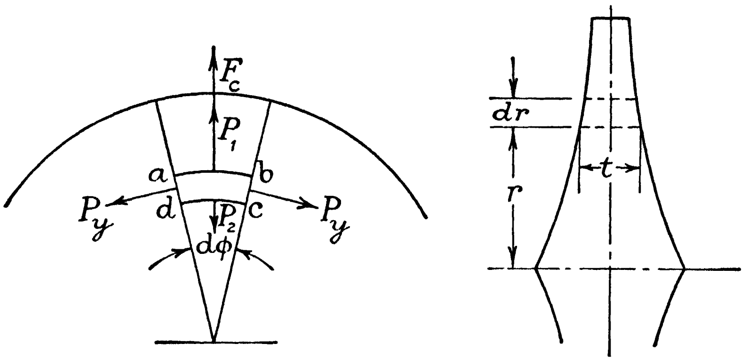


Fig. 12.5

The algebraic sum of the forces on the element *abcd* must be zero for equilibrium. Taking compressive forces as negative and resolving the forces on the element outwards, we have (cf. p. 281)

$$-P_2 + P_1 - 2P_y \sin\tfrac{1}{2}\,d\phi + F_c = 0,$$

or $$-p_x r\,d\phi\,t + (p_x + dp_x)(r + dr)\,d\phi\,(t + dt) - 2p_y t\,dr\,\tfrac{1}{2}d\phi$$

$$+ \frac{w}{g}(r + \tfrac{1}{2}dr)\,dr\,d\phi\,t\omega^2(r + \tfrac{1}{2}dr) = 0.$$

Hence $$tr\,dp_x + tp_x\,dr + p_x r\,dt - p_y t\,dr + \frac{w}{g}\omega^2 r^2 t\,dr = 0,$$

neglecting squares and products of small quantities dp_x, dr and dt.

$$\therefore\ p_y = p_x + r\frac{dp_x}{dr} + p_x\frac{r}{t}\cdot\frac{dt}{dr} + \frac{w}{g}\omega^2r^2.$$

But
$$t = cr^{-\beta},$$

$$\therefore\ \frac{dt}{dr} = -\beta cr^{\ \beta-1}.$$

$$\therefore\ p_y = p_x + r\frac{dp_x}{dr} - p_x\beta + \frac{w}{g}\omega^2r^2$$

$$= \frac{d}{dr}(rp_x) - \beta p_x + \frac{w}{g}\omega^2r^2.$$

$$\therefore\ \frac{d}{dr}(rp_x) = p_y + \beta p_x - \frac{w}{g}\omega^2r^2. \quad . \quad . \quad . \quad . \quad (29)$$

Since the axial stress is neglected,

$$\frac{du}{dr} = \frac{1}{E}\left(p_x - \frac{p_y}{m}\right), \qquad \text{[equation (1)]}$$

and
$$\frac{u}{r} = \frac{1}{E}\left(p_y - \frac{p_x}{m}\right). \qquad \text{[equation (2)]}$$

By differentiating equation (2) with respect to r,

$$\frac{du}{dr} = \frac{d}{dr}\left(\frac{rp_y}{E}\right) - \frac{d}{dr}\left(\frac{rp_x}{mE}\right).$$

But
$$\frac{du}{dr} = \frac{1}{E}\left(p_x - \frac{p_y}{m}\right), \qquad \text{[equation (1)]}$$

and
$$\frac{d}{dr}(rp_x) = p_y + \beta p_x - \frac{w}{g}\omega^2r^2, \qquad \text{[equation (29)]}$$

$$\therefore\ \frac{1}{E}\left(p_x - \frac{p_y}{m}\right) = \frac{d}{dr}\left(\frac{rp_y}{E}\right) - \frac{p_y}{mE} - \frac{\beta p_x}{mE} + \frac{w\omega^2r^2}{gmE}.$$

$$\therefore\ p_x + \frac{\beta p_x}{m} = \frac{d}{dr}(rp_y) + \frac{w}{gm}\omega^2r^2.$$

$$\therefore\ \left(1 + \frac{\beta}{m}\right)p_x = \frac{d}{dr}(rp_y) + \frac{w}{gm}\omega^2r^2. \quad . \quad . \quad . \quad . \quad (30)$$

Multiplying throughout by r, and differentiating with respect to r,

$$\left(1 + \frac{\beta}{m}\right)\frac{d}{dr}(rp_x) = \frac{d}{dr}\left[r\frac{d}{dr}(rp_y) + \frac{w}{gm}\omega^2r^3\right].$$

Writing y for rp_y and substituting for $\frac{d}{dr}(rp_x)$ from (29),

$$\left(1+\frac{\beta}{m}\right)\left(\frac{y}{r}+\beta p_x-\frac{w}{g}\,\omega^2r^2\right)=r\frac{d^2y}{dr^2}+\frac{dy}{dr}+\frac{3w\omega^2r^2}{gm}. \quad (31)$$

Substituting for p_x in (31) from (30),

$$\left(1+\frac{\beta}{m}\right)\left[\frac{y}{r}+\frac{\beta}{1+\beta/m}\frac{d}{dr}(rp_y)+\frac{\beta}{1+\beta/m}\cdot\frac{w\omega^2r^2}{gm}-\frac{w}{g}\,\omega^2r^2\right]$$

$$=r\frac{d^2y}{dr^2}+\frac{dy}{dr}+\frac{3w\omega^2r^2}{gm}.$$

$$\therefore\ \left(1+\frac{\beta}{m}\right)\frac{y}{r}+\beta\frac{dy}{dr}-\frac{w\omega^2r^2}{g}=r\frac{d^2y}{dr^2}+\frac{dy}{dr}+\frac{3w\omega^2r^2}{gm}.$$

$$\therefore\ r\frac{d^2y}{dr^2}+(1-\beta)\frac{dy}{dr}-\left(1+\frac{\beta}{m}\right)\frac{y}{r}=-\frac{w\omega^2r^2}{g}\left(\frac{3}{m}+1\right).$$

Multiplying throughout by r, a homogeneous linear equation is obtained, namely,

$$r^2\frac{d^2y}{dr^2}+(1-\beta)r\frac{dy}{dr}-\left(1+\frac{\beta}{m}\right)y=-\frac{w\omega^2r^3}{g}\left(\frac{3}{m}+1\right). \quad (32)$$

Put $y=r^n$ and equate the left-hand side of equation (32) to zero, i.e.

$$r^2n(n-1)r^{n-2}+(1-\beta)rnr^{n-1}-\left(1+\frac{\beta}{m}\right)r^n=0.$$

$$\therefore\ n(n-1)+(1-\beta)n-\left(1+\frac{\beta}{m}\right)=0,$$

i.e.

$$n^2-\beta n-\left(1+\frac{\beta}{m}\right)=0.$$

$$\therefore\ n=\tfrac{1}{2}\left[\beta\pm\sqrt{\left\{\beta^2+4\left(1+\frac{\beta}{m}\right)\right\}}\right].$$

Let

$$k^2=\beta^2+4\left(1+\frac{\beta}{m}\right),$$

then the complementary function is

$$Ar^{\frac{1}{2}(\beta+k)}+Br^{\frac{1}{2}(\beta-k)},$$

and

$$y=Ar^{\frac{1}{2}(\beta+k)}+Br^{\frac{1}{2}(\beta-k)}=r^{\frac{1}{2}\beta}\left[Ar^{\frac{1}{2}k}+Br^{-\frac{1}{2}k}\right].$$

For the particular integral, substitute Hr^n for y, then

$$r^2 n(n-1)Hr^{n-2} + (1-\beta)rnHr^{n-1} - \left(1+\frac{\beta}{m}\right)Hr^n$$
$$= -\frac{w\omega^2 r^3}{g}\left(\frac{3}{m}+1\right).$$

$$\therefore\ Hr^n\left[n^2 - \beta n - \left(1+\frac{\beta}{m}\right)\right] = -\frac{w\omega^2 r^3}{g}\left(\frac{3}{m}+1\right).$$

$$n = 3,$$

and
$$Hr^3\left[3^2 - 3\beta - \left(1+\frac{\beta}{m}\right)\right] = -\frac{w\omega^2 r^3}{g}\left(\frac{3}{m}+1\right),$$

$$\therefore\ H = -\frac{\frac{w\omega^2}{g}\left(\frac{3}{m}+1\right)}{9 - 3\beta - \left(1+\frac{\beta}{m}\right)}.$$

The complete solution is

$$y = -\frac{\frac{w\omega^2}{g}\left(\frac{3}{m}+1\right)}{9 - 3\beta - \left(1+\frac{\beta}{m}\right)}r^3 + r^{\frac{1}{2}\beta}[Ar^{\frac{1}{2}k} + Br^{-\frac{1}{2}k}].$$

But $y = rp_y$,

$$\therefore\ p_y = -\frac{\frac{w\omega^2}{g}\left(\frac{3}{m}+1\right)r^2}{9 - 3\beta - \left(1+\frac{\beta}{m}\right)} + r^{\frac{1}{2}\beta-1}[Ar^{\frac{1}{2}k} + Br^{-\frac{1}{2}k}].$$

From (30),

$$p_x = \frac{1}{1+\frac{\beta}{m}}\left[\frac{d}{dr}(rp_y) + \frac{w}{mg}\omega^2 r^2\right]$$

$$= \frac{1}{1+\frac{\beta}{m}}\left\{-\frac{\frac{3w\omega^2}{g}\left(\frac{3}{m}+1\right)r^2}{9 - 3\beta - \left(1+\frac{\beta}{m}\right)}\right.$$
$$\left. + r^{\frac{1}{2}\beta-1}\left[(\tfrac{1}{2}\beta + \tfrac{1}{2}k)Ar^{\frac{1}{2}k} + (\tfrac{1}{2}\beta - \tfrac{1}{2}k)Br^{-\frac{1}{2}k}\right] + \frac{w}{gm}\omega^2 r^2\right\}.$$

Let $\beta = 1$ and $1/m = \frac{1}{4}$, then, as $k^2 = 6$,

$$p_x = -\tfrac{13}{19} \cdot \frac{w\omega^2r^2}{g} + 1{\cdot}38Ar^{0\cdot725} - 0{\cdot}58Br^{-1\cdot725},$$

$$p_y = -\tfrac{7}{19}\frac{w\omega^2r^2}{g} + r^{-\frac{1}{2}}(Ar^{\frac{1}{2}\sqrt{6}} + Br^{-\frac{1}{2}\sqrt{6}}).$$

When $r = 0$, p_x and p_y are infinite unless $B = 0$,

$$\therefore\ B = 0.$$

$p_x = 0$ when $r = R$ (external radius),

$$\therefore\ 0 = -\tfrac{13}{19}\frac{w\omega^2R^2}{g} + 1{\cdot}38AR^{0\cdot725}.$$

$$\therefore\ A = +\,0{\cdot}495\,\frac{w}{g}\,\omega^2R^{1\cdot275}.$$

$$\therefore\ p_x = -\,\tfrac{13}{19}\,\frac{w\omega^2r^2}{g} + 0{\cdot}685\,\frac{w\omega^2}{g}\,R^{1\cdot275}r^{\cdot725}$$

$$= \frac{0{\cdot}685w\omega^2}{g}\,[R^{1\cdot275}r^{\cdot725} - r^2].$$

$$p_y = -\tfrac{7}{19} \cdot \frac{w\omega^2r^2}{g} + 0{\cdot}495\,\frac{w}{g}\,\omega^2R^{1\cdot275}r^{\cdot725}$$

$$= \frac{w\omega^2}{g}\,[0{\cdot}495R^{1\cdot275}r^{\cdot725} - 0{\cdot}368r^2].$$

4. Disc uniformly stressed.

When the radial and circumferential stresses are everywhere equal to one another and of constant magnitude, the disc will be of uniform strength.

From equation (4),

$$\frac{d}{dr}(rp_x) = p_y - \frac{w}{g}\,\omega^2r^2.$$

But $p_x = p_y = p$,

$$\therefore\ \frac{d}{dr}(rp) = p - \frac{w}{g}\,\omega^2r^2.$$

If t is the thickness of the disc at radius r,

$$\frac{d}{dr}(trp) = tp - \frac{w}{g}\,\omega^2r^2t.$$

Dividing throughout by p,

$$\frac{d}{dr}(tr) = t - \frac{w}{g}\omega^2 r^2 \frac{t}{p}.$$

$$\therefore \frac{dt}{dr} = -\frac{w}{g}\omega^2 r \frac{t}{p},$$

or

$$\frac{dt}{t} = -\frac{w}{g} \cdot \frac{\omega^2 r}{p} dr.$$

$$\therefore \log_e t = -\frac{w}{g}\frac{\omega^2 r^2}{2p} + A.$$

If t_0 is the thickness when $r = 0$, then

$$\log_e t_0 = A,$$

$$\therefore \log_e t = -\frac{w}{g}\frac{\omega^2 r^2}{2p} + \log_e t_0,$$

or

$$t = t_0 e^{-\frac{w\omega^2 r^2}{2gp}}.$$

EXAMPLES

1. A circular disc, of uniform thickness small compared with its radius, is rotated about its axis with a uniform angular velocity ω. Prove that at any radius r the radial stress p_r and the hoop stress p_0 are:

$$p_r = A + \frac{B}{r^2} - \frac{3+\sigma}{8} \cdot \frac{\rho\omega^2}{g} r^2,$$

$$p_0 = A - \frac{B}{r^2} - \frac{1+3\sigma}{8} \cdot \frac{\rho\omega^2}{g} r^2,$$

where A and B are constants, ρ is the density of the material, and σ is Poisson's ratio.

Deduce expressions for these stresses for an annular disc of inside and outside radii r_1 and r_2 respectively.

2. Show that the maximum intensity of hoop tension in a thin rotating disc is

$$\rho \frac{\omega^2 R^2}{g} \cdot \frac{3m+1}{8m},$$

where ρ is the density of the material, R is the radius of the disc, ω its angular velocity, and $1/m$ is Poisson's ratio for the material. (London Univ.)

3. A thin steel disc is revolved at 4000 rev. per min. The diameter of the disc is 2 ft., the density of the steel 0·28 lb., and Poisson's ratio $\frac{1}{4}$. Find the maximum stress (*a*) when the disc is solid, (*b*) when it has a concentric hole 12 in. diameter, (*c*) when the hole is large compared with the disc diameter, so that the disc becomes a thin ring.

[3·32 tons per sq. in.; 7·04 tons per sq. in.; 8·19 tons per sq. in.]

4. Find the thickness at the axis of a steel disc, 2 ft. 6 in. diameter, so that the stress is uniform everywhere and equal to 5 tons per sq. in. The thickness of the disc at the periphery is 1 in., and the revolutions per minute of the disc are 7500. The density of the steel is 0·28 lb. per c. in.

[1·57 in.]

CHAPTER XIII

Flat Plates

When a flat plate is supported at its edges and loaded transversely, it bends in a similar manner to a beam. The bending does not take place in one plane, consequently lateral stresses must be introduced as well as longitudinal stresses. In the following theory it will be assumed that the neutral surface is unstressed. This assumption is justified so long as the deflection of the plate is small compared with its thickness.

Circular Plate symmetrically loaded

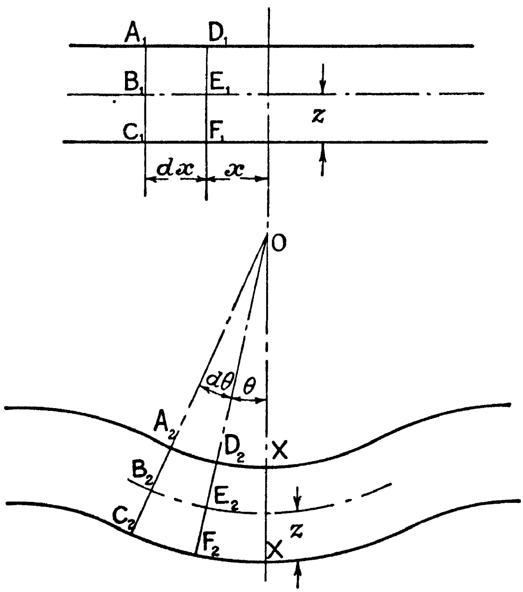


Fig. 13.1

Parallel lines $A_1B_1C_1$, $D_1E_1F_1$ before loading become non-parallel lines $A_2B_2C_2$, $D_2E_2F_2$ after loading.

Let B_1E_1 be points on the neutral surface and let z be the distance

of C_1F_1 from B_1E_1. F_2E_2 produced meets the perpendicular line through the centre of the plate X in O, and is inclined to it at an angle θ. C_2B_2 produced makes an angle $d\theta$ with F_2D_2O.

Before bending, $D_1E_1F_1$ will lie on circles of radius x, and $A_1B_1C_1$ will lie on circles of radius $x + dx$.

After bending, the radii of the circles through C_2 and F_2 will be increased to $(x + dx) + (\theta + d\theta)z$ and $(x + \theta z)$ respectively.

The circumferential strain at radius x, that is, point F_2, is

$$e_1 = \frac{2\pi(x + \theta z) - 2\pi x}{2\pi x}$$

$$= \frac{\theta z}{x}$$

$$= \frac{z}{\rho} \text{ approximately}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ is the radius of curvature in a plane perpendicular to XXO and parallel to XO. The difference in radii between C_1 and F_1 and C_2 and F_2 has increased from dx to $dx + zd\theta$, hence the radial strain

$$e_2 = \frac{dx + zd\theta - dx}{dx} = \frac{zd\theta}{dx}$$

$$= \frac{z}{\rho_1} \text{ approximately}, \quad . \quad . \quad (2)$$

where ρ_1 is the radius of curvature in the plane XXO. From the deflection of beams,

$$\frac{1}{\rho_1} = \frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where y is the deflection.

Now $$\frac{1}{\rho_1} = \frac{d\theta}{dx}, \text{ whilst } \frac{1}{\rho} = \frac{\theta}{x}.$$

$$\therefore \frac{1}{\rho} = \frac{1}{x}\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If f_1 and f_2 are the stresses corresponding to the strains e_1 and e_2,

$$e_1 = \frac{f_1}{E} - \frac{f_2}{mE}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$e_2 = \frac{f_2}{E} - \frac{f_1}{mE}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $1/m$ = Poisson's ratio.

Substituting from (1) and (2) in (5) and (6) for e_1 and e_2,

$$\frac{z}{\rho} = \frac{f_1}{E} - \frac{f_2}{mE}, \quad \ldots \quad (7)$$

$$\frac{z}{\rho_1} = \frac{f_2}{E} - \frac{f_1}{mE}. \quad \ldots \quad (8)$$

Substituting from (3) and (4) for $1/\rho_1$ and $1/\rho$,

$$\frac{z}{x}\frac{dy}{dx} = \frac{f_1}{E} - \frac{f_2}{mE}, \quad \ldots \quad (9)$$

$$z\frac{d^2y}{dx^2} = \frac{f_2}{E} - \frac{f_1}{mE}. \quad \ldots \quad (10)$$

Solving the simultaneous equations (9) and (10), we multiply (10) by $1/m$ and add to (9).

$$\frac{z}{x}\frac{dy}{dx} + \frac{z}{m}\frac{d^2y}{dx^2} = \frac{f_1}{E} - \frac{f_1}{m^2E}.$$

$$\therefore f_1 = \frac{m^2E}{m^2-1}\left(\frac{z}{x}\frac{dy}{dx} + \frac{z}{m}\frac{d^2y}{dx^2}\right). \quad \ldots \quad (11)$$

Multiplying (9) by $1/m$ and adding to (10),

$$\frac{z}{mx}\frac{dy}{dx} + z\frac{d^2y}{dx^2} = \frac{f_2}{E} - \frac{f_2}{m^2E},$$

$$\therefore f_2 = \frac{m^2E}{m^2-1}\left(\frac{z}{mx}\frac{dy}{dx} + z\frac{d^2y}{dx^2}\right). \quad \ldots \quad (12)$$

Equations (11) and (12) cannot be solved until the relationship between y and x is known, but they indicate that f_1 and f_2 are directly proportional to z. Hence the stress is zero at the neutral surface and a maximum at the outer edges of the material.

Consider a semicircular ring on which the stresses f_1 and f_2 act. It has radius x, width dx, thickness dz and is distant z from the neutral surface (see fig. 13.2, p. 306).

Resolving the forces along OX, the algebraic sum must be zero for equilibrium.

The total circumferential force component along OX is

$$2f_1\,dx\,dz.$$

The total outward radial force component along OX is

$$(f_2 + df_2)\ (\text{projected area}) = (f_2 + df_2)(x + dx)2\,dz.$$

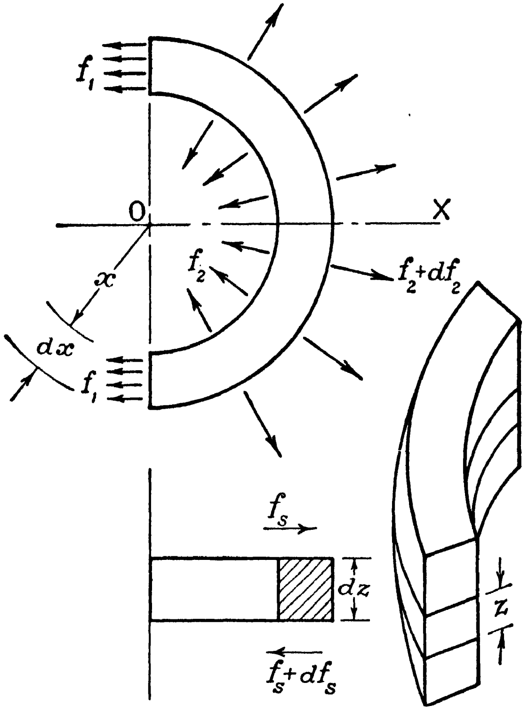


Fig. 13.2

The total inward radial force component along OX is

$$f_2 2x\,dz.$$

If f_s be the shear stress at distance z from the neutral surface, and $f_s + df_s$ the shear stress at distance $(x + dx)$, then the net total shear-stress component along OX is

$$[(f_s + df_s) - f_s] \times (\text{projected area}) = df_s \times 2x\,dx.$$

Hence

$$2f_1\,dx\,dz + f_2 2x\,dz - (f_2 + df_2)(x + dx)2\,dz - df_s 2x\,dx = 0.$$

$$\therefore\ 2f_1\,dx\,dz - 2f_2\,dx\,dz - 2\,df_2 x\,dz - 2\,df_2\,dx\,dz - 2\,df_s x\,dx = 0.$$

Neglecting the term containing the product $df_2\,dx$, since each term is small, and dividing throughout by $2dz$,

$$f_1\,dx - f_2\,dx - df_2 x - df_s x \,.\, \frac{dx}{dz} = 0.$$

$$\therefore\ \frac{f_1}{x} - \frac{f_2}{x} - \frac{df_2}{dx} - \frac{df_s}{dz} = 0.$$

Substituting from (11) and (12) for f_1 and f_2,

$$\frac{df_s}{dz} = -\frac{m^2Ez}{m^2-1}\left[\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right].$$

Integrating with respect to z, since the term inside the brackets is independent of z,

$$f_s = -\frac{m^2Ez^2}{2(m^2-1)}\left[\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right] + A.$$

Known conditions: $f_s = 0$ at top and bottom of plate, i.e. when $z = \pm\frac{1}{2}t$ where t is the thickness of the plate.

$$\therefore\ A = \frac{m^2Et^2}{8(m^2-1)}\left[\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right].$$

$$\therefore\ f_s = \frac{m^2E(t^2-4z^2)}{8(m^2-1)}\left[\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right]. \quad . \quad (13)$$

Before a solution can be effected it is necessary to decide on the type of loading. In most practical cases the load will be distributed uniformly over the surface. Let a distributed load of w units per unit area be spread over the surface.

Uniformly distributed load.

Total load on plate of radius x is

$$\pi x^2 w = \int_{-\frac{1}{2}t}^{+\frac{1}{2}t} 2\pi x\, dz\, f_s.$$

By substituting for f_s from (13) and integrating,

$$\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} = \frac{6(m^2-1)}{m^2Et^3}\,.\,wx. \quad . \quad . \quad (14)$$

Hence from equations (13) and (14),

$$f_s = \frac{m^2E(t^2-4z^2)}{8(m^2-1)}\left[\frac{6(m^2-1)}{m^2Et^3}\,.\,wx\right]$$

$$= \frac{3}{4}\frac{(t^2-4z^2)}{t^3}\,.\,wx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The shear stress is a maximum when $z = 0$, i.e.

$$f_s = \frac{3}{4}\frac{wx}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Equation (14) may be written as

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}\right)=\frac{6(m^2-1)}{m^2Et^3}\,.\,wx.$$

$$\therefore\ \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\int\frac{6(m^2-1)}{m^2Et^3}\,.\,wx\,dx$$

$$=\frac{6(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{2}wx^2+B.$$

Multiplying both sides of the above equation by x,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{6(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{2}wx^3+Bx,$$

or

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)=\frac{6(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{2}wx^3+Bx.$$

$$\therefore\ x\frac{dy}{dx}=\frac{6(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{8}wx^4+\tfrac{1}{2}Bx^2+C.$$

$$\therefore\ \frac{dy}{dx}=\frac{6(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{8}wx^3+\tfrac{1}{2}Bx+\frac{C}{x}. \quad . \quad . \quad . \quad . \quad (15)$$

$$\therefore\ y=\frac{6(m^2-1)}{m^2Et^3}\,.\,\frac{wx^4}{32}+\tfrac{1}{4}Bx^2+C\log x+D. \quad (16)$$

(*a*) *Circular plate freely supported at the edge.*

Known conditions:

$$\frac{dy}{dx}=\text{slope}=0\text{ when }x=0,\ \therefore\ C=0;$$

also $y=0$ when $x=r$,

$$\therefore\ 0=\frac{6(m^2-1)}{m^2Et^3}\,.\,\frac{wr^4}{32}+\tfrac{1}{4}Br^2+D.$$

$$\therefore\ D=-\frac{6(m^2-1)}{m^2Et^3}\,.\,\frac{wr^4}{32}-\tfrac{1}{4}Br^2.$$

The radial stress f_2 will be zero when $x=r$, therefore, substituting in equation (12) for dy/dx and d^2y/dx^2, the value of B will be obtained. Differentiating (15),

$$\frac{d^2y}{dx^2}=\frac{18(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{8}wx^2+\tfrac{1}{2}B.$$

$$\therefore\ f_2=\frac{m^2Ez}{m^2-1}\left[\frac{6(m^2-1)}{m^2Et^3}\,.\,\frac{wx^2}{8m}+\frac{B}{2m}+\frac{18(m^2-1)}{m^2Et^3}\,.\,\tfrac{1}{8}wx^2+\tfrac{1}{2}B\right].$$

Here $f_2 = 0$

$$= \frac{m^2Ez}{m^2-1}\left[\frac{6(m^2-1)wr^2}{8m^3Et^3} + \frac{18(m^2-1)wr^2}{8m^2Et^3} + \frac{B}{2m} + \tfrac{1}{2}B\right].$$

$$\therefore\ B\left(\frac{1}{2m} + \tfrac{1}{2}\right) = -\frac{6(m^2-1)wr^2}{8m^2Et^3}\left(\frac{1}{m} + 3\right).$$

$$\therefore\ B\frac{(1+m)}{2m} = -\frac{6(m^2-1)wr^2}{8m^2Et^3}\left(\frac{1+3m}{m}\right).$$

$$\therefore\ B = -\frac{6(m-1)wr^2(3m+1)}{4m^2Et^3}$$

$$= -\tfrac{3}{2}\frac{(m-1)(3m+1)wr^2}{m^2Et^3}.$$

$$D = -\frac{6(m^2-1)}{m^2Et^3}\cdot\frac{wr^4}{32} + \tfrac{3}{8}\frac{(m-1)(3m+1)}{m^2Et^3}wr^4$$

$$= -\frac{3(m-1)wr^4}{8m^2Et^3}\left[\tfrac{1}{2}(m+1) - (3m+1)\right]$$

$$= \tfrac{3}{16}\frac{(m-1)(5m+1)}{m^2Et^3}\cdot wr^4.$$

Hence

$$y = \frac{6(m^2-1)}{m^2Et^3}\cdot\frac{wx^4}{32} - \tfrac{3}{8}\frac{(m-1)(3m+1)}{m^2Et^3}\cdot wr^2x^2 + \tfrac{3}{16}\frac{(m-1)(5m+1)}{m^2Et^3}\cdot wr^4$$

$$= \tfrac{3}{8}\frac{(m-1)}{m^2Et^3}w\left[\tfrac{1}{2}(m+1)x^4 - (3m+1)r^2x^2 + \tfrac{1}{2}(5m+1)r^4\right]. \quad (17)$$

$$\therefore\ \frac{dy}{dx} = \frac{6(m^2-1)}{m^2Et^3}\cdot\frac{wx^3}{8} - \tfrac{3}{4}\frac{(m-1)(3m+1)}{m^2Et^3}wr^2x$$

$$= \tfrac{3}{4}\frac{(m-1)w}{m^2Et^3}\left[(m+1)x^3 - (3m+1)r^2x\right], \quad (18)$$

and $$\frac{d^2y}{dx^2} = \tfrac{3}{4}\frac{(m-1)w}{m^2Et^3}\left[3(m+1)x^2 - (3m+1)r^2\right]. \quad (19)$$

By substituting in equations (11) and (12) for dy/dx and d^2y/dx^2, the values of f_1 and f_2 may be obtained.

$$f_1 = \frac{m^2Ez}{m^2 - 1}\left\{\frac{3}{4}\frac{(m-1)w}{m^2Et^3}\right\}\left\{(m+1)x^2 - (3m+1)r^2 + \frac{3(m+1)x^2}{m} - \frac{(3m+1)r^2}{m}\right\}$$

$$= \frac{3}{4}\frac{wz}{(m+1)t^3}\left\{(m+1)x^2 + \frac{3(m+1)x^2}{m} - (3m+1)r^2 - (3m+1)\frac{r^2}{m}\right\}$$

$$= \frac{3}{4}\frac{wz}{t^3m}\left[mx^2 + 3x^2 - m\frac{(3m+1)}{m+1}r^2 - \frac{(3m+1)}{m+1}r^2\right]$$

$$= \frac{3}{4}\frac{wz}{t^3m}[(m+3)x^2 - (3m+1)r^2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$f_2 = \frac{m^2Ez}{m^2 - 1}\left\{\frac{3}{4}\frac{(m-1)w}{m^2Et^3}\right\}\left\{\frac{(m+1)}{m}x^2 - \frac{(3m+1)}{m}r^2 + 3(m+1)x^2 - (3m+1)r^2\right\}$$

$$= \frac{3}{4}\frac{wz}{(m+1)t^3}\left\{\frac{(m+1)}{m}x^2 + 3(m+1)x^2 - \frac{(3m+1)}{m}r^2 - (3m+1)r^2\right\}$$

$$= \frac{3}{4}\frac{wz}{t^3m}\left\{x^2 + 3mx^2 - \frac{(3m+1)}{m+1}r^2 - m\frac{(3m+1)}{m+1}r^2\right\}$$

$$= \frac{3}{4}\frac{wz}{t^3m}\left\{x^2(1+3m) - \frac{(3m+1)}{m+1}r^2(1+m)\right\}$$

$$= -\frac{3}{4}\frac{wz}{t^3m}\{(3m+1)(r^2 - x^2)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

f_1 and f_2 are a maximum when $x = 0$ and $z = \pm\frac{1}{2}t$.

$$\text{Maximum } f_1 = \pm\frac{3}{4}\frac{w \cdot \frac{1}{2}t}{t^3m}[(3m+1)r^2]$$

$$= \pm\frac{3}{8}\frac{w}{t^2m}(3m+1)r^2. \quad . \quad . \quad . \quad (22)$$

$$\text{Maximum } f_2 = \pm\frac{3}{8}\frac{w}{t^2m}(3m+1)r^2. \quad . \quad . \quad . \quad (23)$$

Notice that maximum f_1 = maximum f_2.

The values of e_1 and e_2 may be obtained by substituting for f_1 and f_2 in equations (5) and (6).

$$e_1 = -\frac{3}{4}\frac{wz}{t^3mE}\left[(3m+1)r^2 - (m+3)x^2 - \frac{(3m+1)(r^2 - x^2)}{m}\right]$$

$$= -\frac{3}{4}\frac{wz(m^2-1)}{t^3m^2E}\left[\left(\frac{3m+1}{m+1}\right)r^2 - x^2\right]. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$e_2 = -\tfrac{3}{4}\frac{wz}{t^3mE}\left[(3m+1)(r^2-x^2) - \frac{(3m+1)}{m}r^2 + \frac{(m+3)}{m}x^2\right]$$

$$= -\frac{3wz(m^2-1)}{4t^3m^2E}\left[\frac{(3m+1)}{m+1}r^2 - 3x^2\right]. \quad . \quad . \quad . \quad . \quad (25)$$

e_1 and e_2 are a maximum when $z = \pm\tfrac{1}{2}t$ and $x = 0$.

$$e_1 = e_2 = \pm\frac{3w(m^2-1)(3m+1)}{8t^2m^2E(m+1)}r^2$$

$$= \pm\tfrac{3}{8}\frac{w(m-1)(3m+1)}{t^2m^2E}r^2. \quad . \quad . \quad . \quad . \quad (26)$$

(*b*) *Circular plate firmly fixed at its circumference.*

Known conditions: $\frac{dy}{dx} = 0$ when $x = 0$ and $x = r$,

$y = 0$ when $x = r$.

As in Case (*a*), $C = 0$,

and $$D = -\frac{6(m^2-1)}{m^2Et^3}\cdot\frac{wr^4}{32} - \tfrac{1}{4}Br^2.$$

$$\frac{dy}{dx} = 0 = \frac{6(m^2-1)}{m^2Et^3}\cdot\tfrac{1}{8}wr^3 + \tfrac{1}{2}Br.$$

$$\therefore\ B = -\tfrac{3}{2}\frac{(m^2-1)}{m^2Et^3}\cdot wr^2.$$

$$\therefore\ D = -\frac{6(m^2-1)}{m^2Et^3}\cdot\frac{wr^4}{32} + \tfrac{3}{8}\frac{(m^2-1)wr^4}{m^2Et^3}$$

$$= \tfrac{3}{16}\frac{(m^2-1)wr^4}{m^2Et^3}.$$

Hence $$y = \tfrac{3}{8}\frac{(m^2-1)w}{m^2Et^3}\left[\tfrac{1}{2}x^4 - r^2x^2 + \tfrac{1}{2}r^4\right]. \quad . \quad . \quad (27)$$

$$\therefore\ \frac{dy}{dx} = \tfrac{3}{4}\frac{(m^2-1)w}{m^2Et^3}\left[x^3 - r^2x\right], \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and $$\frac{d^2y}{dx^2} = \tfrac{3}{4}\frac{(m^2-1)w}{m^2Et^3}\left[3x^2 - r^2\right]. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

By substituting in equations (11) and (12) for dy/dx and d^2y/dx^2, the values of f_1 and f_2 may be obtained.

$$f_1 = \tfrac{3}{4}\frac{wz}{t^3}\left[x^2 - r^2 + \frac{3x^2}{m} - \frac{r^2}{m}\right]$$

$$= -\tfrac{3}{4}\frac{wz}{mt^3}[(m+1)r^2 - (m+3)x^2]. \quad . \quad . \quad . \quad (30)$$

$$f_2 = \tfrac{3}{4}\frac{wz}{t^3}\left[\frac{x^2 - r^2}{m} + 3x^2 - r^2\right]$$

$$= -\tfrac{3}{4}\frac{wz}{mt^3}[(m+1)r^2 - (3m+1)x^2]. \quad . \quad . \quad (31)$$

$f_1 = f_2$ when $x = 0$ and $z = \pm\frac{1}{2}t$, but the maximum stress is the radial stress f_2 and occurs when $x = r$ and $z = \pm\frac{1}{2}t$.

For $x = 0$, $$f_1 = f_2 = \pm\tfrac{3}{8}\frac{w}{mt^2}[(m+1)r^2]. \quad . \quad . \quad . \quad . \quad (32)$$

$$\text{Maximum stress } f_2 = \pm\tfrac{3}{8}\frac{w}{mt^2}(-2mr^2) = \pm\tfrac{3}{4}\frac{wr^2}{t^2}. \quad (33)$$

The values of e_1 and e_2 may be obtained by substituting for f_1 and f_2 in equations (5) and (6).

$$e_1 = -\tfrac{3}{4}\frac{wz}{mt^3E}\left[(m+1)r^2 - (m+3)x^2 - \frac{(m+1)}{m}r^2 + \frac{(3m+1)}{m}x^2\right]$$

$$= \tfrac{3}{4}\frac{wz}{m^2t^3E}(m^2-1)(r^2 - x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$e_2 = -\tfrac{3}{4}\frac{wz}{mt^3E}\left[(m+1)r^2 - (3m+1)x^2 - \left(\frac{m+1}{m}\right)r^2 + \left(\frac{m+3}{m}\right)x^2\right]$$

$$= -\tfrac{3}{4}\frac{wz}{m^2t^3E}(m^2-1)(r^2 - 3x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$e_1 = e_2$ when $z = \pm\frac{1}{2}t$ and $x = 0$, then

$$e_1 = e_2 = \pm\tfrac{3}{8}\frac{w}{Em^2t^2}(m^2-1)r^2, \quad . \quad . \quad . \quad (36)$$

but the maximum strain is the radial strain e_2 when $x = r$ and $z = \pm\frac{1}{2}t$.

$$\text{Maximum } e_2 = \pm\tfrac{3}{8}\frac{w}{m^2t^2E}(m^2-1)(r^2 - 3r^2)$$

$$= \pm\tfrac{3}{4}\frac{w(m^2-1)r^2}{m^2t^2E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Example 1.—The pressure on a 3 ft. 6 in. diameter plate is 75 lb. per sq. in. and the thickness is 2 in. Plot graphs to show the variations with the radius in the radial and circumferential stresses on the surface of the plate when the plate is simply supported at its edge. What is the value of the maximum deflection? $E = 30 \times 10^6$ lb. per sq. in.; $1/m = \frac{1}{3}$.

Substituting in the equations

$$f_1 = -\frac{3}{4}\frac{wz}{t^3 m}[(3m+1)r^2 - (m+3)x^2],$$

and

$$f_2 = -\frac{3}{4}\frac{wz}{t^3 m}[(3m+1)(r^2 - x^2)],$$

where $w = 75$ lb. per sq. in., $z = \frac{1}{2}t = 1$ in., $1/m = \frac{1}{3}$, $r = 21$ in.

$$\therefore f_1 = -\frac{3}{4}\frac{75 \times 1}{2^3 \times 3}[(3 \times 3 + 1)21^2 - (3+3)x^2]$$

$$= -2{\cdot}34375[4410 - 6x^2]$$

$$= -(10{,}340 - 14{\cdot}06x^2),$$

and

$$f_2 = -\frac{3}{4}\frac{75 \times 1}{2^3 \times 3}[(3 \times 3 + 1)(21^2 - x^2)]$$

$$= 2{\cdot}34375[4410 - 10x^2]$$

$$= (10{,}330 - 23{\cdot}4x^2).$$

x	0	3	6	9	12	15	18	21 in.
f_1	10,340	10,210	9830	9200	8310	7170	5780	4130 lb. per sq. in.
f_2	10,340	10,130	9490	8140	6960	5060	2740	0 lb. per sq. in.

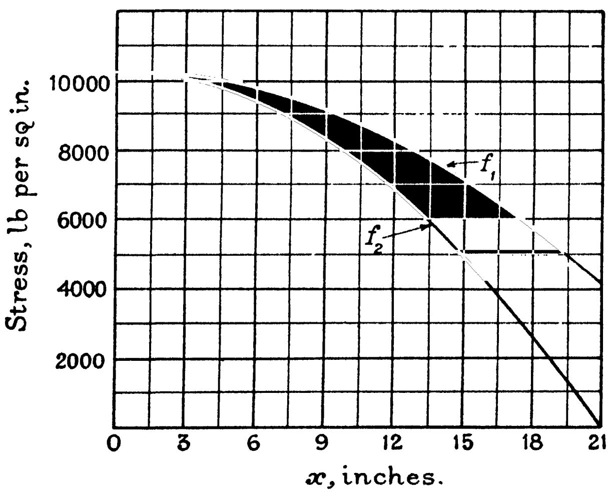


Fig. 13.3

The maximum deflection is given when $x = 0$,

$$y = -\frac{3}{8}\frac{(m-1)w}{m^2Et^3}[\tfrac{1}{2}(5m+1)r^4].$$

$$= -\frac{3}{8}\frac{(3-1)75}{9 \times 30 \times 10^6 \times \frac{1}{2^3}}[\tfrac{1}{2}(5 \times 3 + 1)21^4]$$

$$= -0{\cdot}041 \text{ in.}$$

Example 2.– The end of an engine piston is 12 in. in diameter and ½ in. thick. It is subjected to a pressure of 150 lb. per sq. in. and may be considered to be firmly clamped at its edge. Calculate the factor of safety on the following theories of elastic failure, (i) maximum stress, (ii) maximum strain, (iii) maximum shear stress or stress difference, (iv) maximum strain energy.

The limit of proportionality for the material is 15 tons per sq. in. Poisson's ratio $\frac{1}{3}$. $E = 13{,}000$ tons per sq. in.

By experiment:

(i) Maximum stress = 15 tons per sq. in.

(ii) Maximum strain $= \dfrac{15}{13{,}000} = {\cdot}001154$ in.

(iii) Maximum stress difference or maximum shear stress = 15 tons per sq. in.

(iv) Maximum strain energy $= \frac{1}{2} \times 15 \times {\cdot}001154$

$= {\cdot}00865$ in.-tons

$= 19{\cdot}4$ in.-lb.

For the flat plate:

(i) Maximum stress $= f_2 = \pm\frac{3}{4}\dfrac{wr^2}{t^2}$, [equation (33)]

where $w = 150$ lb. per sq. in., $r = 6$ in., $t = \frac{1}{2}$ in.

$$\therefore f_2 = \pm\frac{3}{4}\frac{150 \times 6^2}{(\frac{1}{2})^2}$$

$= \pm 16{,}200$ lb. per sq. in.

$= \pm 7{\cdot}25$ tons per sq. in.

(ii) Maximum extension $= \dfrac{3}{4}\dfrac{w(m^2-1)r^2}{m^2t^2E}$ [equation (36)]

$$= \frac{3 \times 150(3^2-1)6^2}{4 \times 3^2 \times (\frac{1}{2})^2 \times 13{,}000 \times 2240}$$

$= 0{\cdot}000495$ in.

(iii) Maximum radial stress $f_2 = 7{\cdot}25$ tons per sq. in. The circumferential stress at the outer edge, i.e. where f_2 is a maximum, is

$$f_1 = \pm\frac{3}{4}\frac{wz}{mt^3}[(m+1)r^2 - (m+3)x^2]. \quad . \quad . \quad . \quad . \quad (30)$$

For $z = \frac{1}{4}$ in.,

$$f_1 = \pm\frac{3}{4}\frac{150 \times \frac{1}{4}}{3 \times (\frac{1}{2})^3}[(3+1)6^2 - (3+3)6^2]$$

$$= \pm[75(144 - 216)]$$

$$= \pm[-5400] \text{ lb. per sq. in.}$$

$$= \pm[-2{\cdot}41] \text{ tons per sq. in.}$$

The negative sign indicates opposite type of stress to the one assumed.

Hence maximum stress difference $= 7{\cdot}25 + 2{\cdot}41$

$= 9{\cdot}66$ tons per sq. in.

(iv) Maximum strain energy $= \frac{1}{2E}[f_2^2 + f_1^2 + \frac{2}{m}f_1f_2]$ (see p. 187), for the two-dimensional case in which the stresses are of opposite type.

$$U = \frac{(7{\cdot}25^2 + 2{\cdot}41^2 + \frac{2}{3} \times 7{\cdot}25 \times 2{\cdot}41)}{2 \times 13{,}000}$$

$$= \frac{52{\cdot}56 + 5{\cdot}81 + 11{\cdot}65}{2 \times 13{,}000}$$

$$= {\cdot}0027 \text{ in.-tons}$$

$$= 6{\cdot}04 \text{ in.-lb.}$$

Factor of safety for (i) $= \dfrac{15}{7{\cdot}25}$

$= 2{\cdot}07.$

Factor of safety for (ii) $= \dfrac{{\cdot}001154}{{\cdot}000495}$

$= 2{\cdot}33.$

Factor of safety for (iii) $= \dfrac{15}{9{\cdot}66}$

$= 1{\cdot}55.$

Factor of safety for (iv) $= \dfrac{19{\cdot}4}{6{\cdot}04}$

$= 3{\cdot}22.$

EXAMPLES

1. A thin diaphragm of radius r_0 is encastered horizontally at its circumference and carries a load w per unit area uniformly distributed. Show that the central deflection is

$$\frac{3w(1-\sigma^2)r_0^4}{16Et^3},$$

where t is the thickness of the diaphragm, E is Young's modulus, and σ is Poisson's ratio.

2. A circular plate is of 3 ft. 6 in. diameter and $\frac{3}{4}$ in. thick. It is subjected to a pressure of 10 lb. per sq. in. Find the maximum stress and the maximum deflection when (*a*) the plate is freely supported at its outer edge, (*b*) the plate is firmly clamped at its outer edge. $E = 30 \times 10^6$ lb. per sq. in.; Poisson's ratio $\frac{1}{3}$.

[(*a*) 4·36 tons per sq. in.; 0·1024 in. (*b*) 2·63 tons per sq. in.; 0·0256 in.]

CHAPTER XIV

Vibrations and Whirling of Shafts

The work on strength of materials has been based on the linear connexion between stress and strain. In all cases it has been assumed that the limit of proportionality stress has not been exceeded. By following the same rule with natural or free vibration problems, any disturbance from the position of equilibrium will result in the introduction of a restoring force, and the magnitude of the restoring force will be proportional to the amount of the disturbance. Since the restoring force, which is proportional to the acceleration, is proportional to the amount of the disturbance, the motion of the body will be simple harmonic when the disturbing force is removed.

1. Simple Harmonic Motion.

Simple harmonic motion is motion in a straight line, which is constantly directed towards a point and in which the acceleration is proportional to the distance of the body from the point.

When a point A is travelling uniformly around the circumference of a circle, the projection of the point A on BC, that is, D, moves with simple harmonic motion.

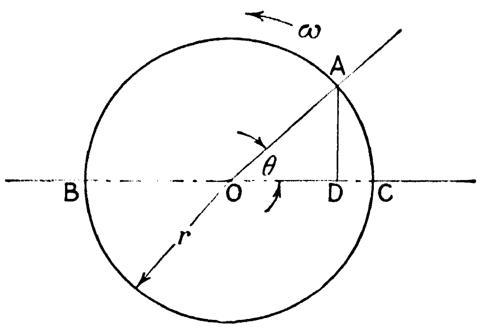


Fig. 14.1

$$\text{The acceleration of A} = \omega^2 r,$$

so that

$$\text{the acceleration of D} = \omega^2 r \cos\theta$$
$$= \omega^2 \,.\, \text{OD},$$

where ω is the angular velocity.

Hence the acceleration of D $\propto$ distance of D from O, which is the condition for D to have simple harmonic motion.

Let t be the time for A to complete one revolution, then $\omega = 2\pi/t$.

But
$$\omega^2 = \frac{\text{acceleration}}{\text{displacement}},$$
$$\therefore \left(\frac{2\pi}{t}\right)^2 = \frac{\text{acceleration}}{\text{displacement}},$$
$$\therefore t = 2\pi\sqrt{\left(\frac{\text{displacement}}{\text{acceleration}}\right)}.$$

Let n be the number of revolutions per second or the frequency of vibration, then
$$n = \frac{1}{t} = \frac{1}{2\pi}\sqrt{\left(\frac{\text{acceleration}}{\text{displacement}}\right)}. \quad . \quad . \quad . \quad . \quad (1)$$

2. Longitudinal Vibrations.

(i) *Helical spring.*—If a heavy weight is supported by a light spring, and the weight is raised and then released, it will descend until the resistance of the spring brings it to rest, i.e. until the restoring force in the spring equals the accelerating force of the falling weight. The motion will be reversed until the compressive resistance of the spring brings the weight to rest. This cycle of operations will continue until gradually damped by frictional resisting forces.

Let e be the stiffness of the spring and x the initial displacement.

Restoring force $= ex$.

Accelerating force $= \frac{W}{g} \times$ acceleration (f),
$$\therefore ex = \frac{W}{g}f.$$
$$\therefore \frac{f}{x} = \frac{ge}{W},$$
i.e.
$$\frac{\text{acceleration}}{\text{displacement}} = \frac{ge}{W}.$$

Substituting in (1),
$$n = \frac{1}{2\pi}\sqrt{\left(\frac{ge}{W}\right)}.$$

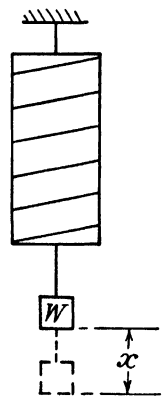


Fig. 14.2

But W/e is the static extension $= \delta$,

$$\therefore\ n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}.$$

Now $$g = 32{\cdot}2 \times 12 \text{ in. per sec. per sec.},$$

$$\therefore\ n = \frac{1}{2\pi}\sqrt{\left(\frac{32{\cdot}2 \times 12}{\delta}\right)} \qquad \text{(where } \delta \text{ is in inches)}$$

$$= \frac{3{\cdot}129}{\sqrt{\delta}}; \qquad \qquad (2)$$

or, $$\text{vibrations per min.} = N$$

$$= \frac{187{\cdot}8}{\sqrt{\delta}}. \qquad \qquad (3)$$

(ii) *Loaded rod.*—When a light rod carries a heavy weight W at the free end, the force necessary to produce an extension x is $P = AEx/L$, where

A = area in sq. in.,

E = modulus of elasticity in lb. per sq. in.,

L = length in inches.

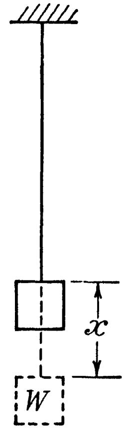


Fig. 14.3

The force P will be equal to the restoring force, so that

$$P = \frac{W}{g}f,$$

or $$\frac{AEx}{L} = \frac{W}{g}f.$$

$$\therefore\ \frac{f}{x} = \frac{AEg}{WL}.$$

Hence $$n = \frac{1}{2\pi}\sqrt{\left(\frac{AEg}{WL}\right)}. \qquad \qquad (4)$$

3. Transverse Vibrations.

Heavy single load on a light shaft.—As with the previous cases, the restoring force will be equal to the acceleration force.

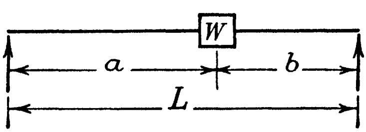


Fig. 14.4

$$\text{Restoring force} = ex.$$

$$\text{Acceleration force} = \frac{W}{g} f.$$

$$\therefore \frac{f}{x} = \frac{ge}{W},$$

and

$$n = \frac{1}{2\pi} \sqrt{\left(\frac{ge}{W}\right)},$$

where $W/e = \delta$ will be the static deflection under the load.

$$\therefore n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}}$$

$$= \frac{3{\cdot}129}{\sqrt{\delta}}, \quad \ldots \ldots \ldots \quad (2)$$

or

$$N = \frac{187{\cdot}8}{\sqrt{\delta}}, \quad \ldots \ldots \ldots \quad (3)$$

as for the helical spring.

The deflection under the load will be given by

$$\delta = \frac{Wa^2b^2}{3EIL}$$

for simply supported ends (see p. 119), or

$$\delta = \frac{Wa^3b^3}{3EIL^3}$$

for fixed ends (see p. 64).

4. Torsional Vibrations.

When a flywheel is fixed to a shaft and the vibration is in circular arcs whose centres lie on the axis of the shaft, the vibration is torsional in nature.

(i) *Single load at one end of a shaft with the other end fixed.*

n = number of oscillations per second (frequency),

θ = amplitude of vibration in radians,

T_0 = torque necessary to produce unit angle of twist,

C = modulus of rigidity,

I = mass moment of inertia of load,

I_0 = polar second moment of area of shaft.

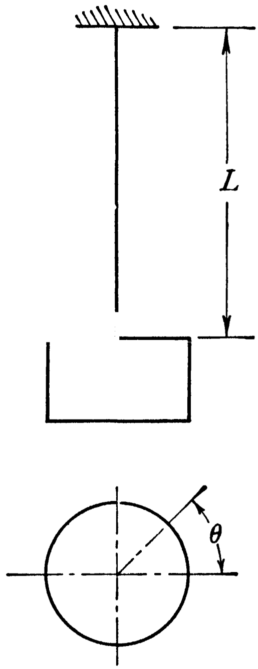


Fig. 14.5

The applied couple is directly proportional to the angular displacement. For amplitude θ the applied couple $= T_0\theta$.

$$\text{Average applied couple} = \tfrac{1}{2}T_0\theta,$$

$$\therefore \text{ work done} = \tfrac{1}{2}T_0\theta \,.\, \theta$$

$$= \tfrac{1}{2}T_0\theta^2.$$

The maximum angular velocity of the load as it passes through the equilibrium position equals the peripheral velocity of the point defining simple harmonic motion, i.e. $2\pi n\theta$.

$$\text{Maximum kinetic energy of load} = \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}I(2\pi n\theta)^2.$$

$$\text{Work done} = \text{change of kinetic energy.}$$

$$\therefore \tfrac{1}{2}T_0\theta^2 = \tfrac{1}{2}I(2\pi n\theta)^2$$

or

$$2\pi n = \sqrt{\frac{T_0}{I}},$$

$$\therefore n = \frac{1}{2\pi}\sqrt{\frac{T_0}{I}}.$$

But

$$\frac{T}{I_0} = \frac{C\theta}{L},$$

and

$$\frac{T}{\theta} = T_0$$

$$= \frac{CI_0}{L}.$$

$$\therefore n = \frac{1}{2\pi}\sqrt{\left(\frac{CI_0}{LI}\right)}. \quad \ldots\ldots \quad (5)$$

(ii) *Two loads.*—When a shaft carries two loads, there will be a node or stationary point somewhere between the two loads. The torsional vibration of the shaft will be the same as that for either load on a shaft fixed at the node. By equating the torsional vibration of W_1 on a shaft of length L_1 to the torsional vibration of W_2 on a shaft of length L_2, it will be possible to find the position of the node, hence the values of L_1 and L_2, and finally the frequency of torsional vibration.

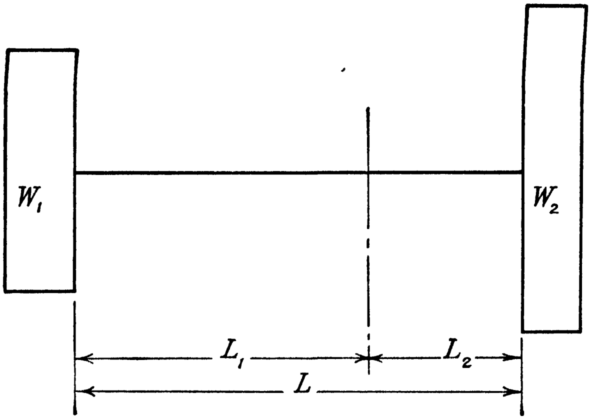


Fig. 14.6

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{CI_0}{L_1 I_1}\right)} = \frac{1}{2\pi}\sqrt{\frac{CI_0}{L_2 I_2}}.$$

$$\therefore \frac{L_1}{L_2} = \frac{I_2}{I_1}.$$

Example 1.—A shaft of 2 in. diameter is freely supported at two points 5 ft. apart and carries a wheel weighing 500 lb. at 3 ft. from the left-hand support. Neglecting the weight of the shaft, calculate the frequency of transverse vibrations. $E = 30 \times 10^6$ lb. per sq. in.

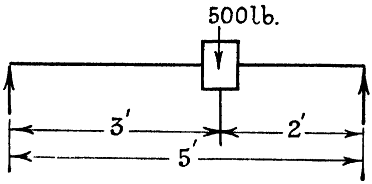


Fig. 14.7

Applying equation (3), $\quad N = \dfrac{187{\cdot}8}{\sqrt{\delta}},$

where $\quad \delta = \dfrac{Wa^2b^2}{3EIL}.$

$W = 500$ lb.,

$a = 3$ ft. $= 36$ in.,

$b = 2$ ft. $= 24$ in.,

$I = \frac{1}{64}\pi \,.\, 2^4 = \frac{1}{4}\pi$ in.^{4} units,

$L = 60$ in.,

$$\therefore\ \delta = \frac{500 \times 36^2 \times 24^2 \times 4}{3 \times 30 \times 10^6 \times \pi \times 60}$$

$$\simeq 0{\cdot}088 \text{ in.,}$$

$$N = \frac{187{\cdot}8}{\sqrt{0{\cdot}088}}$$

$$= 634 \text{ revolutions per minute.}$$

Example 2.—A shaft of 3 in. diameter and 2 ft. long carries a flywheel at one end weighing 2 tons and having a radius of gyration of 2 ft. 6 in. At the other end there is a rope pulley weighing 25 cwt. with a radius of gyration of 15 in. Calculate the frequency of the natural torsional vibration if the modulus of rigidity for the steel shaft is 12 × 10⁶ lb. per sq. in.

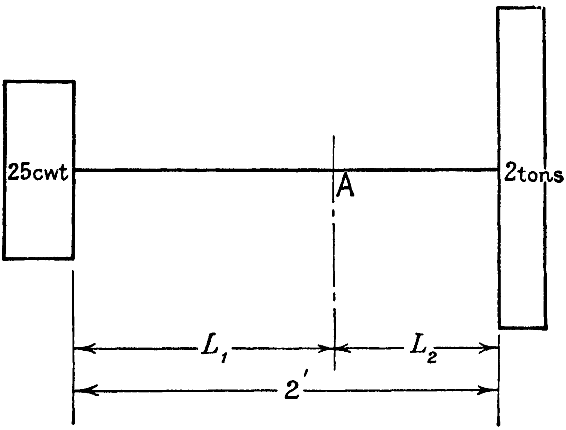


Fig. 14.8

Let A be the node on the shaft, then the frequency of vibration

$$n = \frac{1}{2\pi}\sqrt{\frac{CI_0}{L_1I_1}} = \frac{1}{2\pi}\sqrt{\frac{CI_0}{L_2I_2}}.$$

$$\therefore\ L_1I_1 = L_2I_2.$$

$$\therefore\ L_1 = L_2\frac{I_2}{I_1}$$

$$= (L - L_1)\frac{I_2}{I_1}.$$

Now $\quad I_2 = \frac{2 \times 2240}{32} \times 2{\cdot}5^2$ lb.-ft. units,

and $\quad I_1 = \frac{25 \times 112}{32} \times 1{\cdot}25^2$ lb.-ft. units.

$$\therefore L_1 = (L - L_1)\frac{2 \times 2240 \times 2 \cdot 5^2}{25 \times 112 \times 1 \cdot 25^2}$$
$$= (L - L_1)6 \cdot 4$$
$$= (2 - L_1)6 \cdot 4.$$
$$\therefore L_1 = 1 \cdot 73 \text{ ft.}$$

Frequency of vibration $= \frac{1}{2\pi}\sqrt{\frac{CI_0}{L_1 I_1}}$,

where
$$I_0 = \tfrac{1}{32}\pi d^4 = \tfrac{1}{32}\pi \times 3^4$$
$$= 7 \cdot 95 \text{ in.}^4 \text{ units.}$$
$$L_1 = 1 \cdot 73 \times 12 = 20 \cdot 76 \text{ in.}$$
$$I_1 = \frac{25 \times 112}{32} \times 1 \cdot 25^2 \times 12^2$$
$$= 19{,}700 \text{ lb.-in. units.}$$
$$C = 12 \times 10^6 \text{ lb. per sq. in.}$$
$$\therefore n = \frac{1}{2\pi}\sqrt{\frac{12 \times 10^6 \times 7 \cdot 95}{20 \cdot 76 \times 19{,}700}}$$
$$= \frac{1}{2\pi}\sqrt{233 \cdot 5}$$
$$= 2 \cdot 44 \text{ cycles per sec.}$$

EFFECT OF MASS OF SHAFT

The foregoing problems have all assumed that the load is large compared with the mass of the shaft or rod and therefore that the latter could be neglected. We will now take it into account.

5. Longitudinal Vibrations.

Unloaded rod.—Let the velocity of the free end be v, then the velocity of any section distance x from the free end will be vx/L. For an element dx of the rod, the kinetic energy will be

$$\frac{w\,dx}{2g}\left(\frac{vx}{L}\right)^2,$$

where w is the mass of the rod per unit length.

$$\therefore \text{Total kinetic energy} = \int_0^L \frac{w\,dx}{2g}\left(\frac{vx}{L}\right)^2$$
$$= \frac{wL}{3} \cdot \frac{v^2}{2g}. \qquad (6)$$

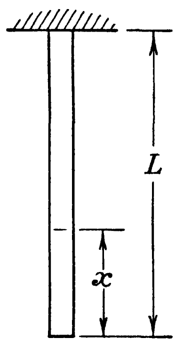


Fig. 14.9

The kinetic energy of a load wL suspended at the free end of the rod would be $wL(v^2/2g)$, hence the mass of the rod may be allowed for by adding one third of its mass to the load attached at its free end.

The same reasoning could be applied to the case of the helical spring, so that the mass of the spring can be allowed for by adding one third of its mass to the suspended weight.

6. Transverse Vibrations.

Unloaded shaft.

The solution is simplified if it is assumed that the curve of flexure when the shaft is vibrating is the same as the static curve of flexure. The calculated frequencies are very nearly equal to one another whether the approximate or exact solution is used.

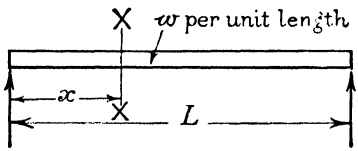


Fig. 14.10

(i) *Approximate solution.*—Let

w = load per unit length,

y = deflection at section XX.

Strain energy of elemental section when in its most deflected position $= \frac{1}{2}wy\,dx$.

$$\text{Strain energy on whole shaft} = \int_0^L \tfrac{1}{2}wy\,dx.$$

$$\text{Maximum velocity of elemental section} = 2\pi ny.$$

$$\text{Kinetic energy of elemental section} = \frac{w\,dx}{2g}(2\pi ny)^2.$$

$$\text{Kinetic energy for whole shaft} = \int_0^L \frac{w}{2g}(2\pi ny)^2\,dx.$$

Equating strain energy to kinetic energy,

$$\int_0^l \tfrac{1}{2}wy\,dx = \int_0^L \frac{w}{2g}(2\pi ny)^2\,dx.$$

$$\therefore\ (2\pi n)^2 = \frac{g\int_0^L y\,dx.}{\int_0^L y^2\,dx}. \qquad (7)$$

In order to solve it is necessary to assume a shape for the curve of flexure, say the static deflection curve.

For any section dx at XX, $EI\dfrac{d^4y}{dx^4} = w.$

$$EI\frac{d^3y}{dx^3} = wx + A.$$

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 + Ax + B.$$

Known conditions: bending moment $= 0$ when $x = 0$ and L.

$$\therefore\ EI\frac{d^2y}{dx^2} = 0 = \tfrac{1}{2}wL^2 + AL + B,$$

and $$EI\frac{d^2y}{dx^2} = 0 = 0 + 0 + B.$$

$$\therefore\ B = 0,$$

and $$A = -\tfrac{1}{2}wL.$$

$$\therefore\ EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 - \tfrac{1}{2}wLx.$$

$$EI\frac{dy}{dx} = \tfrac{1}{6}wx^3 - \tfrac{1}{4}wLx^2 + C.$$

$$EIy = \tfrac{1}{24}wx^4 - \tfrac{1}{12}wLx^3 + Cx + D.$$

Known conditions: $y = 0$ when $x = 0$ and L. Hence

$$D = 0,$$

and $$EIy = 0 = \tfrac{1}{24}wL^4 - \tfrac{1}{12}wL^4 + CL.$$

$$\therefore\ C = \tfrac{1}{24}wL^3.$$

Then $$EIy = \tfrac{1}{24}(wx^4 - 2wLx^3 + wL^3x).$$

$$\therefore\ y = \frac{w}{24EI}(x^4 - 2Lx^3 + L^3x).$$

$$\therefore\ y^2 = \left(\frac{w}{24EI}\right)^2\{x^8 + 4L^2x^6 + L^6x^2 - 4Lx^7 + 2L^3x^5 - 4L^4x^4\}.$$

Hence $$\int_0^L y\,dx = \frac{w}{24EI}\cdot\frac{L^5}{5},$$

and $$\int_0^L y^2\,dx = \left(\frac{w}{24EI}\right)^2 L^9\left(\tfrac{31}{630}\right).$$

Substituting in (7),

$$(2\pi n)^2 = \frac{g\,\dfrac{w}{24EI}\cdot\dfrac{L^5}{5},}{\left(\dfrac{w}{24EI}\right)^2 L^9 \times \frac{31}{630}}$$

$$= g\cdot\frac{24EI}{wL^4}\times\tfrac{126}{31}.$$

The maximum deflection for a simply supported beam with a uniformly distributed load is

$$\frac{5wL^4}{384EI} = \delta.$$

$$\therefore\ \frac{wL^4}{EI} = \tfrac{384}{5}\delta.$$

Hence $(2\pi n)^2 = g \times 24 \times \dfrac{5}{384\delta} \times \tfrac{126}{31}$

$$= 1{\cdot}27\,\frac{g}{\delta},$$

or $$n = \sqrt{1{\cdot}27} \times \frac{3{\cdot}129}{\sqrt{\delta}} \quad (\text{taking } g = 12 \times 32{\cdot}2 \text{ in./sec.}^2)$$

$$= 1{\cdot}127 \times \frac{3{\cdot}129}{\sqrt{\delta}} \text{ per sec.} \qquad (8)$$

$$N = 1{\cdot}127 \times \frac{187{\cdot}8}{\sqrt{\delta}} \text{ per min.}$$

$$= \frac{211{\cdot}7}{\sqrt{\delta}} \text{ per min.} \qquad (9)$$

(ii) *Exact solution.*—For an element of length dx the displacement from the undeflected position is y when the shaft is revolving, hence the centrifugal force on the element dx is $\frac{w\,dx}{g}\,\omega^2 y$. It is therefore seen that the force on the shaft varies with y, and consequently the static deflection curve of flexure for the shaft is not applicable.

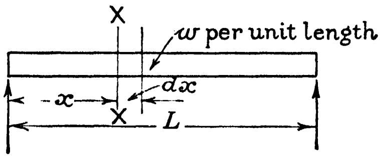


Fig. 14.11

The load at XX $= \frac{w}{g}\,\omega^2 y.$

From beam theory, $$EI\frac{d^4y}{dx^4} = \frac{w}{g}\,\omega^2 y, \qquad (10)$$

or $$\frac{d^4y}{dx^4} = k^4 y,$$

where $$k^4 = \frac{w\omega^2}{gEI}. \qquad (11)$$

The solution of the differential equation

$$\frac{d^4y}{dx^4} - k^4 y = 0$$

is $$y = A\cos kx + B\sin kx + C\cosh kx + D\sinh kx, \qquad (12)$$

where the constants A, B, C and D may be found from the known conditions.

Known conditions: (i) $y = 0$ when $x = 0$ and L,

(ii) bending moment $= 0$ when $x = 0$ and L,

i.e. $$EI\frac{d^2y}{dx^2} = 0 \text{ when } x = 0 \text{ and } L.$$

When $x = 0$,

$$y = 0 = A\cos 0 + B\sin 0 + C\cosh 0 + D\sinh 0$$
$$= A + C. \qquad (a)$$

When $x = L$,

$$y = 0 = A\cos kL + B\sin kL + C\cosh kL + D\sinh kL. \qquad (b)$$

By differentiating equation (12) twice,

$$\frac{d^2y}{dx^2} = -k^2A\cos kx - k^2B\sin kx + k^2C\cosh kx + k^2D\sinh kx.$$

When $x = 0$,

$$\frac{d^2y}{dx^2} = 0 = -k^2A\cos 0 - k^2B\sin 0 + k^2C\cosh 0 + k^2D\sinh 0$$
$$= -k^2A + k^2C.$$

As k^2 is not zero, then $\quad 0 = -A + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$

When $x = L$,

$$\frac{d^2y}{dx^2} = 0 = -k^2A\cos kL - k^2B\sin kL + k^2C\cosh kL + k^2D\sinh kL. \quad (d)$$

Solving for A and C by equations (a) and (c), we have

$$A = C = 0.$$

Equations (b) and (d) now become

$$0 = B\sin kL + D\sinh kL, \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$
$$0 = -k^2B\sin kL + k^2D\sinh kL,$$

but k^2 is not zero,

$$\therefore\ 0 = -B\sin kL + D\sinh kL. \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Adding (b) and (d), $\quad 0 = 2D\sinh kL,$

but $\sinh kL$ cannot be zero, $\quad \therefore\ D = 0.$

Hence from (b), $\quad B\sin kL = 0.$

In this case B cannot be zero, otherwise there is no solution,

$$\therefore\ \sin kL = 0.$$
$$\therefore\ kL = \pi,\ 2\pi,\ 3\pi,\ \&c.,$$

and $\quad k = \frac{\pi}{L},\ \frac{2\pi}{L},\ \frac{3\pi}{L},\ \&c.$

Substituting $k = \frac{\pi}{L}$ in (11),

$$\left(\frac{\pi}{L}\right)^4 = \frac{w\omega^2}{gEI}.$$

$$\therefore\ \omega^2 = \frac{\pi^4}{L^4}\frac{gEI}{w},$$

or

$$\omega = \pi^2\sqrt{\left(\frac{gEI}{wL^4}\right)}.$$

But the maximum static deflection $= \frac{5}{384}\frac{wL^4}{EI}$,

and

$$\frac{wL^4}{EI} = \frac{384\delta}{5}.$$

Hence

$$\omega = \pi^2\sqrt{\left(\frac{5g}{384\delta}\right)}.$$

But

$$2\pi n = \omega,$$

$$\therefore\ n = \frac{\pi}{2}\sqrt{\frac{5g}{384\delta}}$$

$$= \frac{3{\cdot}52}{\sqrt{\delta}} \text{ per sec.},$$

or

$$N = \frac{211{\cdot}2}{\sqrt{\delta}} \text{ per min.} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It will be seen that the values of N given by equations (9) and (13) are approximately equal, and therefore the assumption made in the derivation of (9) is justified. It should be noticed that equation (13) was found when using the least value of k, consequently there are other frequencies than $N = 211{\cdot}2/\sqrt{\delta}$ per minute at which the shaft vibrates, namely, 4, 9, &c., times the least or fundamental frequency.

7. Torsional Vibrations.

Unloaded shaft.—Let the angular velocity of the free end be ω, then the angular velocity of any section distance x from the free end will be $\omega x/L$.

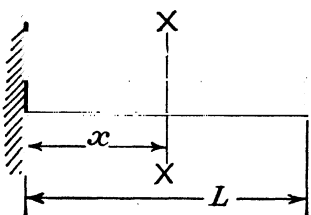


Fig. 14.12

If I_s be the mass moment of the complete shaft, then the mass moment of an element dx will be

$$\frac{dx}{L} \cdot I_s,$$

and the kinetic energy will be

$$\tfrac{1}{2}\left(\frac{dx\,I_s}{L}\right)\left(\frac{x\omega}{L}\right)^2 = \frac{I_s\omega^2}{2L^3}(x^2\,dx).$$

$$\therefore \text{ Total kinetic energy} = \int_0^L \frac{I_s\omega^2}{2L^3}x^2\,dx$$

$$= \tfrac{1}{6}I_s\omega^2.$$

Therefore the kinetic energy of the unloaded shaft is equivalent to the kinetic energy of a load of mass moment $\frac{1}{3}I_s$ placed at the free end of the shaft. Hence the mass of the shaft may be allowed for by adding one third of its mass moment to that of the load on the shaft.

8. Transverse Vibration of a Shaft with Several Loads.

If a shaft carries several loads, the frequency of vibration may be found by substituting the frequencies of vibration, when the shaft carries any one of the loads alone, into the following empirical formula:

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2} + \dots + \frac{1}{N_s^2},$$

where N = frequency of total system,

N_1 = frequency when load 1 is acting by itself,

N_2 = frequency when load 2 is acting by itself,

N_s = frequency of the shaft under its own mass.

Substituting for N_1, N_2, &c., from equation (3), where δ_1, δ_2, &c., are the static deflections, the equation becomes

$$\frac{1}{N^2} = \frac{\delta_1}{187{\cdot}8^2} + \frac{\delta_2}{187{\cdot}8^2} + \frac{\delta_3}{187{\cdot}8^2} + \dots + \frac{\delta_s}{211{\cdot}2^2}.$$

Example 3.—A 2 in. diameter mild-steel rod is freely supported over a span of 100 in. The rod carries a 50 lb. weight at the centre of the span and a 30 lb. weight placed 10 in. from each support. Find the frequency of transverse vibration. Density of mild steel is 0·28 lb. per c. in. $E = 30 \times 10^6$ lb. per sq. in.

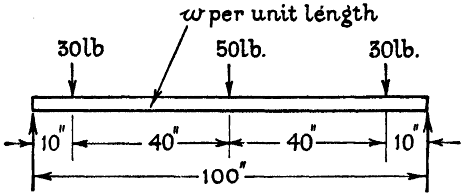


Fig. 14.13

It will be necessary to find the static deflections produced by the separate loadings.

Deflection produced by weight of rod $\delta_s = \frac{5}{384}\frac{WL^3}{EI}$,

where
$$W = \tfrac{1}{4}\pi \times 2^2 \times 100 \times \cdot 28 = 88 \text{ lb.},$$
$$I = \tfrac{1}{64}\pi \times 2^4 = 0\cdot 7854 \text{ in.}^4 \text{ units},$$
$$L = 100 \text{ in.},$$
$$E = 30 \times 10^6 \text{ lb. per sq. in.}$$
$$\therefore \delta_s = \frac{5 \times 88 \times 100^3}{384 \times 30 \times 10^6 \times \cdot 7854}$$
$$= \cdot 0486 \text{ in.}$$
$$\therefore N_s = \frac{211\cdot 2}{\sqrt{\cdot 0486}}$$
$$= 960 \text{ per min.}$$

Deflection under 50 lb. weight produced by 50 lb. weight alone
$$= \delta_1 = \frac{WL^3}{48EI},$$
where
$$W = 50 \text{ lb.}$$
$$\therefore \delta_1 = \frac{50 \times 100^3}{48 \times 30 \times 10^6 \times \cdot 7854}$$
$$= \cdot 0442 \text{ in.}$$
$$N_1 = \frac{187\cdot 8}{\sqrt{\delta_1}}$$
$$= \frac{187\cdot 8}{\sqrt{\cdot 0442}}$$
$$= 890 \text{ per min.}$$

Deflection under 30 lb. weight produced by 30 lb. weight alone

$$= \delta_2 = \frac{Wa^2b^2}{3EIL},$$

where $W = 30$ lb.,

$a = 10$ in.,

$b = 90$ in.,

$L = 100$ in.

$$\therefore \delta_2 = \frac{30 \times 10^2 \times 90^2}{3 \times 30 \times 10^6 \times \cdot 7854 \times 100}$$

$$= \cdot 003435 \text{ in.}$$

$$N_2 = \frac{187 \cdot 8}{\sqrt{\delta_2}}$$

$$= \frac{187 \cdot 8}{\sqrt{\cdot 003435}}$$

$$= 3200 \text{ per min.}$$

The frequency of vibration for the second 30 lb. weight N_3 will be the same, namely, 3200 per min. Combining the frequencies in order to ascertain the actual frequency of the system by the empirical formula

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2} + \frac{1}{N_s^2},$$

$$\frac{1}{N^2} = \frac{1}{890^2} + \frac{1}{3200^2} + \frac{1}{3200^2} + \frac{1}{960^2}$$

$$= 1 \cdot 262 \times 10^{-6} + \cdot 098 \times 10^{-6} + \cdot 098 \times 10^{-6} + 1 \cdot 085 \times 10^{-6}$$

$$= 2 \cdot 543 \times 10^{-6}.$$

$$\therefore N^2 = 393{,}300,$$

$$\therefore N = 626 \text{ per minute.}$$

9. Whirling of Shafts.

It will be impossible to have a shaft in which the centre of gravity coincides with the axis of rotation; consequently, a centrifugal force will act on the shaft when it is rotated, and this will cause further deviation of the centre of gravity from the axis of rotation.

Fig. 14.14 represents a shaft of negligible mass which is carrying a load W.

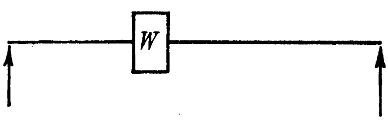


Fig. 14.14

Let h = initial displacement of centre of gravity from the axis of rotation,

y = further displacement of centre of gravity when the shaft is running at ω radians per second,

e = stiffness of shaft, i.e. the force which will produce unit deflection of the shaft.

Centrifugal force at ω radians per second $= \frac{W}{g}\omega^2(h + y)$.

Restoring force at that speed $= ey$.

Equating, since system is in equilibrium,

$$ey = \frac{W}{g}\omega^2(h + y).$$

$$\therefore\ y\left(e - \frac{W}{g}\omega^2\right) = \frac{W}{g}\omega^2 h.$$

$$\therefore\ y = \frac{\frac{W}{g}\omega^2 h}{e - \frac{W}{g}\omega^2}$$

$$= \frac{h}{\frac{ge}{W\omega^2} - 1}.$$

When $ge/(W\omega^2) = 1$, y is infinite. The value of ω which will make $ge/(W\omega^2) = 1$ is known as the critical or whirling speed.

If $$\frac{ge}{W\omega^2} = 1,$$

then $$\omega = \sqrt{\frac{ge}{W}}$$

$$= \sqrt{\frac{g}{\delta}},$$

where $$\delta = \text{static deflection} = \frac{W}{e}.$$

Now $$\omega = 2\pi n,$$

$$\therefore\ n = \frac{1}{2\pi}\omega$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$= \frac{3{\cdot}129}{\sqrt{\delta}}, \text{ where } \delta \text{ is in inches.}$$

Hence the critical speed corresponds to the natural frequency of vibration [cf. equation (2)].

If $ge/(W\omega^2)$ is less than 1, i.e. if ω is very large, then the value of y will become less, and the shaft will straighten out; thus, at a particular speed of rotation, the centre of gravity will coincide with the axis of rotation.

Unloaded shaft.

Let y = deflection of shaft at XX when it is revolving at ω radians per second,

w = load per unit length.

$$\text{Centrifugal force} = \frac{w}{g}\omega^2 y.$$

From beam theory,

$$EI\frac{d^4y}{dx^4} = \frac{w}{g}\omega^2 y. \qquad (14)$$

Now this equation is equation (10), which was used to find the frequency of transverse vibration of an unloaded shaft. Consequently, the critical or whirling speed coincides with the frequency of transverse vibration. There will be whirling speeds other than the lowest or fundamental as was seen in connexion with the frequency of transverse vibrations. The lowest whirling speed will be

$$n = \frac{3{\cdot}52}{\sqrt{\delta}} \text{ per sec.,}$$

or

$$N = \frac{211{\cdot}2}{\sqrt{\delta}} \text{ per min. (see p. 330).}$$

Example 4.—The shaft of a De Laval steam turbine is of $\frac{3}{8}$ in. diameter, and is freely supported in short bearings over a span of 12 in. A wheel, weight 4 lb., is placed midway between the bearings. Calculate the critical speed (*a*) of the unloaded shaft, (*b*) for the wheel alone, i.e. neglect the weight of the shaft, (*c*) for the combined system.

Density of shaft material is 0·28 lb. per c. in. $E = 30 \times 10^6$ lb. per sq. in.

(*a*) For the shaft, $\delta_s = \frac{5}{384}\frac{WL^3}{EI}$,

where $W = \frac{1}{4}\pi(\frac{3}{8})^2 \times 12 \times \cdot 28 = 0\cdot 37$ lb.,

$L = 12$ in.,

$I = \frac{1}{64}\pi(\frac{3}{8})^4$

$= \cdot 000972$ in.^{4} units.

$$\therefore \delta_s = \frac{5 \times \cdot 37 \times 12^3}{384 \times 30 \times 10^6 \times \cdot 000972}$$

$$= \cdot 000285 \text{ in.}$$

$$N_s = \frac{211\cdot 2}{\sqrt{\cdot 000285}}$$

$$= 12{,}500 \text{ per minute.}$$

(*b*) For the wheel, $\delta_1 = \frac{W_1 L^3}{48EI}$,

where $W_1 = 4$ lb.

$$\therefore \delta_1 = \frac{4 \times 12^3}{48 \times 30 \times 10^6 \times \cdot 000972}$$

$$= \cdot 00494 \text{ in.}$$

$$N_1 = \frac{187\cdot 8}{\sqrt{\delta_1}}$$

$$= \frac{187\cdot 8}{\sqrt{\cdot 00494}}$$

$$= 2670 \text{ per minute.}$$

(*c*) For the combined system,

$$\frac{1}{N^2} = \frac{1}{N_s^2} + \frac{1}{N_1^2}$$

$$= \frac{1}{12{,}500^2} + \frac{1}{2670^2}$$

$$= \cdot 640 \times 10^{-8} + 14\cdot 03 \times 10^{-8}.$$

$$\therefore N^2 = 6\cdot 82 \times 10^6,$$

$$\therefore N = 2600 \text{ per minute.}$$

'haft subjected to end thrust.

If a rotating shaft has an end thrust the deflection of the shaft ill be increased.

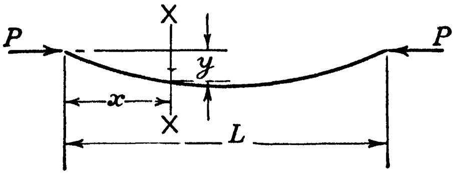


Fig. 14.15

Let the end thrust on a shaft be P, then there will be a bending oment Py at XX which will be due to the end thrust. Hence

$$EI\frac{d^2y}{dx^2} = -Py.$$

y differentiating this expression twice,

$$EI\frac{d^4y}{dx^4} = -P\frac{d^2y}{dx^2}.$$

ombining this with equation (14),

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} - \frac{w\omega^2 y}{g} = 0,$$

$$\therefore \frac{d^4y}{dx^4} + \frac{P}{EI}\frac{d^2y}{dx^2} - \frac{w\omega^2 y}{gEI} = 0. \tag{15}$$

The solution of this equation may be found when the equation

$$m^4 + \frac{P}{EI}m^2 - \frac{w\omega^2}{gEI} = 0$$

as been solved.

The values of m for such an equation must be

$$\pm\sqrt{\left[-\frac{P}{2EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}}\right]}$$

'nd

$$\pm\sqrt{\left[-\frac{P}{2EI} - \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}}\right]}.$$

ıe complete solution is

$$y = A\cos m_1x + B\sin m_1x + C\cosh m_2x + D\sinh m_2x,$$

where $$m_1^2 = -\frac{P}{2EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}},$$

and $$m_2^2 = -\frac{P}{2EI} - \sqrt{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}}.$$

Known conditions: if the shaft is freely supported at its ends, then $y = 0$ and $d^2y/dx^2 = 0$ when $x = 0$ and L.

Substituting $x = 0$ in the equations for y and d^2y/dx^2 gives $A = 0$ and $C = 0$ (compare pp. 328–9). Hence

$$y = B \sin m_1x + D \sinh m_2x,$$

and $$\frac{d^2y}{dx^2} = -m_1^2B \sin m_1x + m_2^2D \sinh m_2x.$$

Substituting $x = L$,

$$y = 0 = B \sin m_1L + D \sinh m_2L, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\frac{d^2y}{dx^2} = 0 = -m_1^2B \sin m_1L + m_2^2D \sinh m_2L. \quad . \quad . \quad (b)$$

Multiplying (a) by m_2^2 and subtracting (b) from (a),

$$0 = m_2^2B \sin m_1L + m_1^2B \sin m_1L$$
$$= (m_2^2 + m_1^2)B \sin m_1L.$$

Now $(m_2^2 + m_1^2)$ is not zero and, if $B = 0$, then there is no problem, consequently

$$\sin m_1L = 0.$$

$$\therefore \; m_1L = \pi, 2\pi, 3\pi, \&c.$$

The lowest speed is given by $m_1 = \frac{\pi}{L}$.

$$\therefore \; m_1^2 = -\frac{P}{2EI} + \sqrt{\left\{\left(\frac{P}{2EI}\right)^2 + \frac{w\omega^2}{gEI}\right\}} = \frac{\pi^2}{L^2}.$$

EXAMPLES

1. A uniform beam of mass 20 lb. per ft. is simply supported at its ends over a span of 10 ft. Determine the lowest natural frequency of transverse vibration of the beam, assuming that the deflected form of the

beam is proportional to the static deflection. The relevant moment of inertia of the section is 6·59 in.^{4}, and the Young's modulus of the material 30×10^6 lb. per sq. in. (London Univ.) [1400 per min.]

✓2. Obtain an expression for the natural frequency of transverse vibration of a bar of length L carrying a uniformly distributed load w lb. per ft., and simply supported at the ends. Assume that the vibration deflection curve is similar to the static deflection curve.

Determine the natural frequency of a beam of I section, 20 ft. long, simply supported at the ends, which carries a uniformly distributed load of 500 lb. per ft. $I_{XX} = 345$ in.^{4} $E = 30 \times 10^6$ lb. per sq. in. (London Univ.) [508 per min.]

✓3. The propeller and engine flywheel of a ship have moments of inertia of 70 and 90 tons-ft.^{2} respectively, and are mounted 40 ft. apart on a hollow shaft 15 in. outside and 12 in. inside diameter. Calculate the frequency of torsional oscillations of the shaft. $C = 12 \times 10^6$ lb. per sq. in. [1·27 per sec.]

✓ 4. The crankshaft of an oil-engine is equivalent to a straight shaft of 4 in. diameter and 4 ft. long with a flywheel at each end. The weight of each flywheel is 2000 lb., and the radius of gyration 2 ft. 9 in. Calculate the frequency of the natural torsional vibrations, taking the modulus of rigidity for the shaft as 12×10^6 lb. per sq. in. [7·18 per sec.]

✓ 5. A shaft of ½ in. diameter is supported in two short bearings 15 in. apart. Midway between the bearings the shaft carries a disc, the plane of which is at right angles to the axis of the shaft. The slope of the shaft at the bearings may be considered unconstrained. Find the approximate speed at which the shaft will whirl if the disc weighs 20 lb. $E = 30 \times 10^6$ lb. per sq. in. [1418 r.p.m.]

✓ 6. Deduce an expression for the critical or whirling speed of a shaft carrying a single concentrated load at any point along its length. The wheel of an impulse turbine weighs 100 lb. and is attached to a shaft of 1½ in. diameter and 18 in. long at a point 7 in. from one end. The shaft is simply supported in the bearings. Determine the critical speed. [4900 r.p.m.]

✓7. Explain what is meant by the critical speed of a shaft.

Determine the critical speed of a shaft of 2 in. diameter and 10 ft. long due to its own weight, assuming the shaft supported by two bearings. How will this speed be affected by a number of pulleys attached to the shaft? The density of steel is 0·28 lb. per c. in. [665 per min.]

✓ 8. A steel shaft, 7½ in. diameter and 8 ft. long, carries three discs weighing 500, 1000, and 450 lb. These may be taken as concentrated loads at distances of 2 ft., 5 ft. and 6 ft. 6 in. respectively from one of the bearings, which form simple supports for the shaft. Find the critical speed, given that the deflections under the loads when acting singly are ·006 in., ·0095 in. and ·0055 in. respectively. The density of steel is 0·28 lb. per c. in. [1230 per min.]

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